

On the subsystems of topological Markov chains

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Abstract. Let S_A be an irreducible and aperiodic topological Markov chain. If $S_{\bar{A}}$ is an irreducible and aperiodic topological Markov chain, whose topological entropy is less than that of S_A , then there exists an irreducible and aperiodic topological Markov chain, whose topological entropy equals the topological entropy at $S_{\bar{A}}$, and that is a subsystem of S_A . If $\bar{S}$ is an expansive homeomorphism of the Cantor discontinuum, whose topological entropy is less than that of S_A , and such that for every $j \in \mathbb{N}$ the number of periodic points of least period j of $\bar{S}$ is less than or equal to the number of periodic points of least period j of S_A , then $\bar{S}$ is topological conjugate to a subsystem of S_A .

Consider an irreducible and aperiodic topological Markov chain. Represent this chain as the shift S_A on a shift space X_A given by a transition matrix A over a finite state space Σ ,

$$\begin{aligned} A(\sigma, \sigma') &\in \{0, 1\}, \quad \sigma, \sigma' \in \Sigma, \\ X_A &= \{(x_i)_{i \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}} : A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}\}, \\ S_A x &= (x_{i+1})_{i \in \mathbb{Z}} \quad (x = (x_i)_{i \in \mathbb{Z}} \in X_A). \end{aligned}$$

We use the notation

$$Z(a) = \{x \in \Sigma^{\mathbb{Z}} : x_i = a_i, 0 \leq i < I\}, \quad a \in \Sigma^{[0, I]}, I \in \mathbb{N},$$

and

$$\mathcal{A}[S_A, I] = \{a \in \Sigma^{[0, I]} : A(a_i, a_{i+1}) = 1, 0 \leq i < I\}, \quad I \in \mathbb{N}$$

and, more generally, given a subshift (X, S) of $\Sigma^{\mathbb{Z}}$, we denote

$$\mathcal{A}[S, I] = \{a \in \Sigma^{[0, I]} : X \cap Z(a) \neq \emptyset\}, \quad I \in \mathbb{N}.$$

Given another irreducible and aperiodic topological Markov chain $S_{\bar{A}}$ such that the entropy condition

$$h(S_A) > h(S_{\bar{A}})$$

is satisfied, the question arises if $S_{\bar{A}}$ is topologically conjugate to the restriction of S_A to a closed S_A -invariant subset of X_A . The answer is here not always affirmative, since the presence in $S_{\bar{A}}$ of periodic points of low period can be an obstruction to isomorphically imbedding $S_{\bar{A}}$ into S_A . In fact, as we shall see in theorem 3, where we characterize the subsystems of S_A , this is the only obstruction. However, it is

always possible, as we shall see in theorems 1 and 2, to find an irreducible and aperiodic topological Markov chain that is a subsystem of S_A and has topological entropy equal to $h(S_{\bar{A}})$. The existence of such a chain is significant in connection with certain problems of noiseless coding theory, that deal with a situation where digital data to be recorded or transmitted is to be encoded so as to conform to the design of the device used. (Problems that arise in such a context are e.g. described in [3, §§ IV, V].) From the existence of such a chain and a theorem of Adler & Marcus [1] it follows that $S_{\bar{A}}$ can be almost homeomorphically imbedded into S_A .

The proofs require a few preparations. First we choose $\alpha, \beta, \omega \in \Sigma$, $\alpha \neq \beta$, $\alpha \neq \omega$,

$$A(\beta, \alpha) = A(\beta, \omega) = 1,$$

together with a block

$$a = (a_i)_{0 \leq i < N} \in \mathcal{A}[S_A, N],$$

where

$$a_0 = \alpha, \quad a_{N-1} = \beta,$$

and where the symbol β appears only once in a . Further, let $\gamma \in \Sigma$, $A(\omega, \gamma) = 1$. Let $S_{A,M}$ be the subshift of finite type that arises by excluding from the system (X_A, S_A) the block $a^{(*)M}$ (* stands for concatenation). Denote

$$\mathcal{C}_M[I] = \{c \in \mathcal{A}[S_{A,M}, I] : c_0 = \gamma, c_{I-1} = \beta\}.$$

LEMMA 1. *For every $\delta > 0$ there are $M, I \in \mathbb{N}$ such that*

$$|\mathcal{C}_M[I']| > e^{(h(S_A) - \delta)I'}, \quad I' > I.$$

Proof. Let L be such that

$$\log A^{L-2}(\sigma, \sigma') > \left(h(S_A) - \frac{\delta}{2}\right)L, \quad \sigma, \sigma' \in \Sigma$$

and let M be such that

$$NM > L.$$

One has $\mathcal{C}_M[I'] \supset \mathcal{B}[I']$, where

$$\mathcal{B}[I'] = \{(b_i)_{0 \leq i < I'} \in \mathcal{A}[S_A, I'] : b_0 = \gamma, b_{I'-1} = \beta, b_{kL-1} = \beta, b_{kL} = \omega, 0 \leq k < I'L^{-1}\}.$$

If

$$I' > 2Lh(S_A)\delta^{-1}$$

then

$$|\mathcal{B}[I']| > e^{(h(S_A) - \delta)I'}.$$

□

We first turn our attention to an irreducible and aperiodic topological Markov chain $S_{\bar{A}}$ where $h(S_A) > h(S_{\bar{A}})$. Let $\bar{A}$ be over a state space $\bar{\Sigma}$, and choose (going to a two-block system if necessary) a $\bar{\rho} \in \bar{\Sigma}$ such that $\bar{A}(\bar{\rho}, \bar{\rho}) = 0$. For $I \in \mathbb{N}$, where $\bar{A}^I$ has all entries positive, we are going to describe an irreducible and aperiodic subshift of finite type $T_{\bar{A}, I}$ as the restriction of the shift to a shift-invariant closed subset $Y_{\bar{A}, I}$ of the shift space

$$((\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}) \cup (\bar{\Sigma} \times \{0\})^{\mathbb{Z}}.$$

We stipulate that only those two-blocks $((\bar{\sigma}, \tau), (\bar{\sigma}', \tau'))$ appear as subblocks of sequences in $Y_{\bar{A}, I}$ where $\bar{A}(\bar{\sigma}, \bar{\sigma}') = 1$, and we stipulate that all $(I+2)$ -blocks $(c_i)_{0 \leq i \leq I+1}$ that appear as subblocks of the sequences in $Y_{\bar{A}, I}$ conform to one of the following eight specifications. These are designed to ensure the irreducibility and aperiodicity of $T_{\bar{A}, I}$ and also to ensure that the set

$$\bar{F} = \{y \in Y_{\bar{A}, I} : y_0 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}\}$$

satisfies

$$\bar{F} \cap T_{\bar{A}, I}^i \bar{F} = \emptyset, \quad 0 < i < I,$$

and

$$Y_{\bar{A}, I} = \bigcup_{0 \leq i < I} T_{\bar{A}, I}^i \bar{F}.$$

- (1) $c_0 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
 $c_i \in \bar{\Sigma} \times \{0\}, \quad 0 < i < I$,
 $c_I \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
 $c_{I+1} \in \bar{\Sigma} \times \{0\}$.
- (2) $c_0 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
 $c_i \in \bar{\Sigma} \times \{0\}, \quad 0 < i < I$,
 $c_I = (\bar{\rho}, 0)$,
 $c_{I+1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$.
- (3) $c_0 = (\bar{\rho}, 0)$,
 $c_1 = (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
 $c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i \leq I$,
 $c_{I+1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$.
- (4) $c_0 = (\bar{\rho}, 0)$,
 $c_1 = (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
 $c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i \leq I$,
 $c_{I+1} = (\bar{\rho}, 0)$.
- (5) There is a j , $1 < j < I$, such that
 $c_i \in \bar{\Sigma} \times \{0\}, \quad 0 \leq i < j-1$,
 $c_{j-1} = (\bar{\rho}, 0)$,
 $c_j \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
 $c_i \in \bar{\Sigma} \times \{0\}, \quad j < i \leq I+1$.
- (6) There is a $\bar{\sigma} \in \bar{\Sigma} - \{\bar{\rho}\}$ such that
 $c_0 = (\bar{\sigma}, 0), \sum_{\bar{\alpha} \neq \bar{\rho}} \bar{A}^{I-1}(\bar{\alpha}, \bar{\sigma}) > 0$,
 $c_1 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
 $c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i \leq I$,
 $c_{I+1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$.

(7) There is a $\bar{\sigma} \in \bar{\Sigma} - \{\bar{\rho}\}$ such that

$$c_0 = (\bar{\sigma}, 0), \sum_{\bar{\alpha} \neq \bar{\rho}} \bar{A}^{I-1}(\bar{\alpha}, \bar{\sigma}) > 0,$$

$$c_1 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\},$$

$$c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i \leq I,$$

$$c_{I+1} = (\bar{\rho}, 0).$$

(8) There is a j , $1 \leq j \leq I$, and a $\bar{\sigma} \in \bar{\Sigma}$ such that

$$c_0 = (\bar{\sigma}, 0), \sum_{\bar{\alpha} \neq \bar{\rho}} A^{I-j+1}(\bar{\alpha}, \bar{\sigma}) > 0$$

$$c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i < j-1,$$

$$c_{j-1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{0\},$$

$$c_j \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\},$$

$$c_i \in \bar{\Sigma} \times \{0\}, \quad j < i \leq I+1.$$

THEOREM 1. *The topological entropy of $T_{\bar{A},I}$ equals the topological entropy of $S_{\bar{A}}$.*

Proof. Let $I' \geq I+2$. An inspection of the above eight rules shows that for every

$$(\bar{\sigma}_i)_{0 \leq i < I'} \in \mathcal{A}[S_{\bar{A}}, I']$$

there is at least one

$$(\bar{\sigma}_i, \tau_i)_{0 \leq i < I'} \in \mathcal{A}[T_{\bar{A},I}, I']$$

and also, that there are at most $I+1$ such elements of $\mathcal{A}[T_{\bar{A},I}, I']$. □

Remark that the proof of this theorem can also be formulated by saying that the mapping that drops the second components is an at most $I+1$ to 1 continuous mapping of $Y_{\bar{A},I}$ onto $X_{\bar{A}}$.

Our next theorem can be obtained as a corollary of theorems 1 and 3. In presenting the following proof, we hope to illustrate the method that is again to be used in the proof of theorem 3.

THEOREM 2. *For I sufficiently large, $T_{\bar{A},I}$ is topologically conjugate to a subsystem of $S_{\bar{A}}$.*

Proof. By lemma 1 we can find $M, I \in \mathbb{N}$ such that

$$|\mathcal{C}_M[I' - NM - 1]| > |\mathcal{A}[S_{\bar{A}}, I']|, \quad I' \geq I.$$

We can then have one-to-one into mappings

$$\phi_I : \mathcal{A}[S_{\bar{A}}, I] \rightarrow \mathcal{C}_M[I - NM - 1],$$

and

$$\phi_{I+1} : \mathcal{A}[S_{\bar{A}}, I+1] \rightarrow \mathcal{C}_M[I - NM].$$

We form the block

$$r = a^{(*)M} * (\omega).$$

Observe that r cannot overlap itself properly, and define a one-to-one continuous and shift-invariant mapping Φ of $Y_{\bar{A},I}$ into $X_{\bar{A}}$ where Φ carries a $\bar{y} \in Y_{\bar{A},I}$ into the

$x \in X_A$ that is determined by requiring

$$\begin{aligned}(x_i)_{0 \leq i < I} &= r * \phi_I((\bar{\sigma}_i)_{0 \leq i < I}), \\ \bar{y}_i &= (\bar{\sigma}_i, \tau_i), \quad 0 \leq i < I \\ \bar{y}_0, \bar{y}_I &\in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\},\end{aligned}$$

and

$$\begin{aligned}(x_i)_{0 \leq i \leq I} &= r * \phi_{I+1}((\bar{\sigma}_i)_{0 \leq i \leq I}), \\ \bar{y}_i &= (\bar{\sigma}_i, \tau_i), \quad 0 \leq i \leq I, \\ \bar{y}_0, \bar{y}_{I+1} &\in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}.\end{aligned}$$

□

We now turn our attention to an expansive homeomorphism S of the Cantor discontinuum. We assume S given as a subshift (X, S) of some full shift space, and we denote by $\mathcal{N}_S[J, K]$, $K \geq J$, the set of all $a \in \mathcal{A}[S, K]$ such that there is for every j , $0 < j < J$, at least one k , $0 \leq k < K - j$ such that

$$a_k \neq a_{k+j}.$$

We have the following lemma of Kakutani–Rohlin type.

LEMMA 2. For all $J, K \in \mathbb{N}$, $K \geq J$, there is a closed open set $F \subset X$ such that

$$F \cap S^j F = \emptyset, \quad 0 < j < J$$

and

$$\bigcup_{-J < j < J} S^j F \supset \bigcup_{a \in \mathcal{N}_S[J, K]} X \cap Z(a).$$

Proof. Enumerate

$$\{Z(a) : a \in \mathcal{N}_S[J, K]\} = \{C_l : 1 \leq l \leq L\},$$

and then obtain inductively an increasing sequence F_l , $1 \leq l \leq L$, of closed open sets by

$$\begin{aligned}F_1 &= C_1 \\ F_{l+1} &= F_l \cup \left(C_{l+1} \cap \left(X - \bigcup_{-J < j < J} S^j F_l \right) \right) \quad 1 < l < L.\end{aligned}$$

Set $F = F_L$.

□

In the sequel Π_j will indicate the set of periodic points of a homeomorphism that has least period j .

THEOREM 3. Let S be an irreducible and aperiodic topological Markov chain, and let $\bar{S}$ be an expansive homeomorphism of the Cantor discontinuum such that

$$h(S) > h(\bar{S})$$

and

$$|\Pi_j(S)| \geq |\Pi_j(\bar{S})|, \quad j \in \mathbb{N}, \quad (1)$$

Then $\bar{S}$ is topologically conjugate to a subsystem of S .

Proof. We let $\bar{S}$ be given as a subshift of some shift space $\bar{X}$, and we continue to let the topological Markov chain to be given as S_A . In view of lemma 1 we have

an $M \in \mathbb{N}$ such that

$$h(S_{A,M}) > h(\bar{S}). \quad (2)$$

There is then an $L \in \mathbb{N}$ such that

$$|\Pi_j(S_{A,M})| \geq |\Pi_j(\bar{S})|, \quad j > L. \quad (3)$$

Let now $P, Q \in \mathbb{N}$ be such that all entries of A^{NP} are positive, and such that

$$Q > P + L + M \quad (4)$$

and form the blocks

$$\begin{aligned} s &= a^{(*)P} * (\omega), \\ t &= a^{(*)Q} * (\omega). \end{aligned}$$

By lemma 1, and by (2), there is a $J \in \mathbb{N}$, $J > (2P + Q)N + 3$ such that

$$|\mathcal{C}_M[J' - (2P + Q)N - 3]| > |\mathcal{A}[\bar{S}, J']|, \quad J' \geq J. \quad (5)$$

Denote

$$\begin{aligned} \mathcal{P}_j &= \{b \in \mathcal{A}[S_A, j]: Z(b) \cap \Pi_j(S_A) \neq \emptyset\}, & 0 < j < L, \\ \mathcal{P}_j &= \{b \in \mathcal{A}[S_{A,M}, j]: Z(b) \cap \Pi_j(S_{A,M}) \neq \emptyset\}, & L \leq j < J, \\ \bar{\mathcal{P}}_j &= \{\bar{b} \in \mathcal{A}[\bar{S}, j]: Z(\bar{b}) \cap \Pi_j(\bar{S}) \neq \emptyset\}, & 0 < j < J. \end{aligned}$$

By (1) and (3) one can assign in a one-to-one manner to every $\bar{b} \in \bar{\mathcal{P}}_j$ a $b \in \mathcal{P}_j$, $0 < j < J$, such that, if

$$(\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{j-1}) \rightarrow (b_0, b_1, \dots, b_{j-1}),$$

then

$$(\bar{b}_{j-1}, \bar{b}_0, \dots, \bar{b}_{j-2}) \rightarrow (b_{j-1}, b_0, \dots, b_{j-2}), \quad 0 < j < J.$$

By (5) one can also have one-to-one into mappings

$$\psi_j: \mathcal{A}[\bar{S}, j] \rightarrow \mathcal{C}_M[j - (2P + Q)N - 3], \quad J \leq j < 2J.$$

Finally we select for all $\bar{b} \in \bar{\mathcal{P}}_j$, $0 < j < J$, blocks

$$s_-(b), s_+(b) \in \mathcal{A}[S_A, NP + 1]$$

such that

$$\begin{aligned} (s_-(b))_{NP} &= \beta, & (s_+(b))_0 &= \gamma, \\ A(\bar{b}_{j-1}, s_-(b)_0) &= A((s_+(b))_{NP}, \bar{b}_0) = 1. \end{aligned}$$

There is a $K \in \mathbb{N}$, $K > J^2$, such that

$$|\mathcal{A}[\bar{S}, K] - \mathcal{N}_{\bar{S}}[J, K]| = \sum_{0 < j \leq J} |\Pi_j(\bar{S})|,$$

for if no such K existed, then it would follow by a compactness argument that for some j , $0 < j \leq J$, $\bar{S}$ had more than $|\Pi_j(\bar{S})|$ periodic points of least period j . We apply now lemma 3 to obtain a closed open set $F \subset \bar{X}$ such that

$$F \cap S^i F = \emptyset, \quad 0 < i < J,$$

$$\bigcup_{-J < i < J} S^i F \supset \bigcup_{\bar{d} \in \mathcal{N}_{\bar{S}}[J, K]} \bar{X} \cap Z(\bar{d}).$$

Define then an $\tilde{F}$ by setting

$$\begin{aligned} F_- &= \left(\bigcap_{-2J < i < 0} (\bar{X} - \bar{S}^i F) \right) \cap F, \\ F_+ &= F \cap \left(\bigcap_{0 < i < 2J} (\bar{X} - S^{-i} F) \right) \\ \tilde{F} &= \bar{S}^J F_- \cup F \cup S^{-J} F, \end{aligned}$$

and have

$$\begin{aligned} \tilde{F} \cap \bar{S}^i \tilde{F} &= \emptyset, \quad 0 < i < J, \\ \bigcap_{-J < i < J} (\bar{X} - S^i \tilde{F}) &\subset \bigcup_{0 < j < J} \bigcup_{\bar{d} \in \bar{\mathcal{P}}_j} \bigcap_{-J < i < J} \bar{X} \cap S^i Z(\bar{d}). \end{aligned} \quad (6)$$

Thus one can define a continuous shift invariant mapping Ψ of $\bar{X}$ into X_A where Ψ carries an $\bar{x} \in \bar{X}$ into the $x \in X_A$ that is determined by the following rules:

(1) If

$$\bar{x} \in Z(\bar{b}) \cap \bigcap_{-J < i < J} (\bar{X} - \bar{S}^i \tilde{F}), \quad \bar{b} \in \bar{\mathcal{P}}_j, \quad 0 < j < J$$

then

$$(x_i)_{0 \leq i < j} = b.$$

(2) If

$$\begin{aligned} \bar{x} \in \left(\bigcap_{-2J < i < 0} (\bar{X} - \bar{S}^i \tilde{F}) \right) \cap S^j Z(\bar{b}) \cap \tilde{F} \cap \left(\bigcap_{0 < i < I_+} (\bar{X} - \bar{S}^{-i} \tilde{F}) \right) \cap \bar{S}^{I_+} \tilde{F}, \\ J < I_+ < 2J, \quad \bar{b} \in \bar{\mathcal{P}}_j, \quad 0 < j \leq J, \end{aligned}$$

then

$$(x_i)_{-(2J-1)j^{-1}j \leq i < I_+} = b^{(*)[2J-1]j^{-1}} * s_-(b) * t * s * \psi((\bar{x}_i)_{0 \leq i < I_+}).$$

(3) If

$$\begin{aligned} \bar{x} \in S^{I_-} \tilde{F} \cap \left(\bigcap_{-I < i < 0} (\bar{X} - \bar{S}^i \tilde{F}) \right) \cap \tilde{F} \cap Z(b) \cap \left(\bigcap_{0 < i < 2J} (\bar{X} - \bar{S}^{-i} \tilde{F}) \right), \\ J < I_- < 2J, \quad \bar{b} \in \bar{\mathcal{P}}_j, \quad 0 < j \leq J, \end{aligned}$$

then

$$(x_i)_{-I_- < i \leq [(2J-1)j^{-1}j]} = \psi((\bar{x}_i)_{-I_- < i \leq 0}) * s * t * s_+(b) * b^{(*)[2J-1]j^{-1}}.$$

(4) If

$$\begin{aligned} \bar{x} \in S^{I_-} \tilde{F} \cap \left(\bigcap_{-I_- < i < 0} (X - S^i \tilde{F}) \right) \cap \tilde{F} \cap \left(\bigcap_{0 < i < I_+} (\bar{X} \cap \bar{S}^{-i} \tilde{F}) \right) \cap \bar{S}^{I_+} \tilde{F}, \\ J < I_-, \quad I_+ < 2J, \end{aligned}$$

then

$$(x_i)_{-I_-+L' \leq i \leq I_+} = \psi((\bar{x}_i)_{-I_- \leq i < 0}) * s * t * s * \psi((\bar{x}_i)_{0 \leq i < I_+}).$$

The structure of the blocks s and t and (4) show that $x \in F$ if and only if

$$\Psi x \in S_A^{(2P+Q)N+3} Z(t).$$

Since we have (6) and since the assignments that entered into the construction of Ψ are one-to-one we conclude that $\Psi\bar{x}$ uniquely determines $\bar{x}$. $\square$

There is a version of the finite generator theorem for ergodic measure preserving transformations of finite entropy, that realizes such a transformation by means of an invariant probability measure of any irreducible and aperiodic topological Markov chain, whose topological entropy exceeds the entropy of the transformation ([4], [2 § 28]). One can say that a corollary of theorem 3 achieves for minimal expansive homeomorphisms of the Cantor discontinuum what the finite generator theorem does for measure preserving transformations.

Corollary. Let S be an irreducible and aperiodic topological Markov chain and let $\bar{S}$ be a minimal expansive homeomorphism of the Cantor discontinuum such that

$$h(S) > h(\bar{S}).$$

Then $\bar{S}$ is topologically conjugate to a subsystem of S_A .

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