

On the least prime in an arithmetic progression II. The Deuring—Heilbronn phenomenon

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In this section we shall give the complete proof of the theorem which was stated in the part I of this paper (we shall quote it under the index [I]):

Theorem. *If $l \in [1, D-1]$; $(l, D) = 1$, then for the least prime $p_{\min}(l, D)$ in the progression $Dx + l$ we have the estimates*

$$\lim_{D \rightarrow \infty} \frac{\ln p_{\min}(l, D)}{\ln D} < \infty, \quad (1)$$

$$p_{\min}(l, D) < D^{C_0} \quad (2)$$

C_0 being an absolute constant.

In [I] the estimate (2) was proved only for one half of l 's for each D , namely for such l 's for which the exclusive real character $\chi \pmod{D}$ (if present), equals to (-1) (cf. [I], § 30). We shall prove also a «weak asymptotic law» analogous to that of [I] § 30, (98) and valid from small primes in all progressions.

§ 1

In [I], § 3 was introduced the positive constant c_0 which may be called «Page—Landau constant» [7] such that there are no zeros of any $L(s, \chi)$ $\pmod{D}$ in the rectangle $1 \geq \sigma \geq 1 - \frac{c_0}{\ln D}$, $|t| \leq D$, except, possibly, one real zero β_0 of a series $L(s, \chi)$ belonging to an «exclusive» real character $\chi \pmod{D}$. If such an «exclusive character» $\chi \pmod{D}$ does not exist, then the «basic theorem» alone leads to the following asymptotic law (cf. [I], § 30, (98)).

If $S(N, l) = \sum_{n \equiv l}^{1, \dots, \infty} \Delta(n) e^{-n/N}$; $x = \ln N$, then

$$\int_M^{2M} dx \int_x^{2x} \frac{S(N, l)}{N} dx = \frac{1}{\varphi(D)} \cdot \frac{3M^2}{2} \left(1 + 2\theta \exp \left(-c_1 \frac{M}{\ln D} \right) + \frac{\theta}{\ln D} \right)$$

provided $M \geq B_2 \ln D$.

Hence (2) follows immediately.

Thus, to prove (2) without limitations, we may assume that the exclusive character $\chi \pmod{D}$ does really exist.

§ 2

The character $X(n)$ may be a non-primitive character (mod D). In this case [4], we transform it into a primitive real character $X_1(n)$ to a modulus D_1/D .

The series $L(s, X_1)$ will have the same zeros, as $L(s, X)$ in $\sigma > 0$, since we have [4]

$$L(s, X_1) = \prod_{p|D} \left(1 - \frac{X(p)}{p^s}\right) L(s, X).$$

By a lemma by A. Walfisz [5] we have

$$X_1(n) = \left(\frac{\eta D_1}{n}\right),$$

where $\eta = +1$ or -1 , ηD_1 is a fundamental discriminant, and $\left(\frac{\eta D_1}{n}\right)$ is Kronecker's symbol. The behaviour of the real zero β_0 of the series $L(s, X_1)$ is well known to be connected with the class-number theory of the corpus $k(\sqrt{\eta D_1})$ the famous Siegel's estimate [6]

$$1 - \beta_0 > C_\varepsilon D_1^{-\varepsilon}; \quad \varepsilon > 0$$

arbitrary, belongs to the subject, but we shall not use it here, because C_ε cannot be effectively calculated. We shall use the weaker estimates by A. Page [7]

$$1 - \beta_0 > \frac{c_1}{\sqrt{D_1} \ln^3 D_1}, \quad (3)$$

$$L(1, X_1) > \frac{c_2}{\sqrt{D_1} \ln^3 D_1}. \quad (4)$$

§ 3

M. Deuring [8] (1933) and H. Heilbronn [9] (1934) have discovered an exceedingly important fact which we shall call here «Deuring—Heilbronn phenomenon». Suppose that there exists a sequence of exclusive characters (mod D_1) for which the zero β_0 draws very near to the point $s=1$. Then it proves that the zeros of all L -series to all the moduli, say $\leq D_1^2$ for $|t| \leq D$ recede from the line $\sigma=1$ and approach the critical line $\sigma=\frac{1}{2}$, except, of course, the zero β_0 . We shall now give a more precise statement of the fact.

Second basic theorem («Deuring—Heilbronn phenomenon»)

Let β_0 be the exclusive zero (for the modulus D). Let $\sigma_0 = 1 - \beta_0$, $\mu_0 = \frac{1}{\sigma_0 \ln D} > 1$. Then the rectangle

$$1 \geq \sigma \geq 1 - c_3 \frac{\ln c_4 \mu_0}{\ln D}, \quad |t| \leq \ln^3 D$$

contains no zeros of any $L(s, \chi)$ to any modulus $\leq D^2$ (in particular, to the modulus D) except the zero β_0 of $L(s, X_1)$.

§ 4

As we shall see, here again the «density—property» [I], (d) is of great importance. If it holds for the series $L(s, \chi)$ with the moduli $D' < D^2$,

then the second basic theorem is rather easy to prove by the methods of [3]. So the main difficulty consists in the avoiding the property (d) which requires complicated computations. We consider now all the L -series with primitive non-principal characters to all the moduli $D' \leq D^2$. They have no zeros in $|t| \leq D$, $\sigma \geq 1 - \frac{c_0}{2 \ln D}$ except the zero β_0 (cf. Page [7], Lemma 9). We consider now the rectangle

$$\frac{1}{2} \leq \sigma \leq 1 - \frac{c_0}{2 \ln D}, \quad |t| \leq \ln^3 D.$$

There exist one or several series $L(s, \chi)$ whose zeros in this rectangle have the largest real part. We take one of them, say $L(s, X_2)$ with a primitive $X_2 \pmod{D_2}$, $D_2 < D^2$. It has a zero $\rho_1 = \beta_1 + it_1$ with $|t_1| \leq \ln^3 D$ such that if any of our $L(s, \chi)$'s has a zero $\rho'_1 = \beta'_1 + it'_1$ with $|t'_1| \leq \ln^3 D$, $\beta'_1 \neq \beta_0$, then $\beta'_1 \leq \beta_1$. We denote $1 - \beta_1 = \frac{\Psi(D)}{\ln D}$ (cf. [I], § 2).

The function $X_1(n)X_2(n)$ is a character $\pmod{D_1D_2}$. There are two different cases in its behaviour: it may be a non-principal or a principal character (the last case may occur only if $X_1(n) = X_2(n)$). We form now the «Faltung» of the series $L(s, X_1)$ and $L(s, X_2)$ —the series $L(s, X_1X_2)$, and introduce the function

$$F(s) = L(s, X_2) \cdot L(s + \sigma_0, X_1X_2), \quad (5)$$

where $\sigma_0 = 1 - \beta_0$. We remark that $F(s)$ is an integer function. This is trivial if X_1X_2 is non-principal, and follows from the fact, that $L(s, X_2)$ has a simple zero $s = \beta_0$ and $L(s + \sigma_0, X_1X_2)$ a simple pole $s = \beta_0$ if $X_1 = X_2$. In this, case $F(\beta_0) \neq 0$. We have also $D_1D_2 \leq D^3$.

§ 5

We introduce the function

$$f(s) = \frac{L'}{L}(s, X_2) + \frac{L'}{L}(s + \sigma_0, X_1X_2) = \frac{F'}{F}(s). \quad (6)$$

It will play here a rôle analogous to that of a the function with the same notation in my papers «On Dirichlet's L -series and prime-number sums» [I] and [II].

We have for $\sigma > 1$

$$f(s) = - \sum_{n=1}^{\infty} \frac{a_n \Lambda(n)}{n^s}, \quad (7)$$

where

$$a_n = X_2(n) + X_1X_2(n)n^{-\sigma_0} = \begin{cases} X_2(n)(1 - n^{-\sigma_0}) & \text{for } X_1(n) = -1, \\ X_2(n)(1 + n^{-\sigma_0}) & \text{for } X_1(n) = +1, \\ X_2(n) & \text{for } X_1(n) = 0. \end{cases} \quad (8)$$

§ 6

The function $f(s)$ has no poles inside the rectangle $1 - \frac{\Psi(D)}{\ln D} \leq \sigma \leq 1$, $|t| \leq \ln^3 D$ and has one or several poles on the left vertical side of it

(cf. § 4). We take such a pole, $\rho_1 = \beta_1 + it_1$, $\beta_1 = 1 - \frac{\Psi(D)}{\ln D}$, $|t_1| \leq \ln^2 D$. We may assume $\frac{\Psi(D)}{\ln D} \leq \frac{1}{3}$. In fact, in view of (3), $\ln \nu_0 < \frac{3}{4} \ln D$ and if $\frac{\Psi(D)}{\ln D} > \frac{1}{3}$, then the second basic theorem is proved with $c_2 = \frac{1}{4}$. We shall now introduce some useful notations: $\mathcal{G}$ will denote the sequence of all the numbers which are products of a square ≥ 1 and a «quadrat-frei» number whose prime factors p imply the unity value to the character X_1 : $X_1(p) = +1$. The set $\mathcal{G}$ has a well known arithmetical meaning in $k(\sqrt{\eta D_1})$, we may also define it as a set of all numbers n for which the coefficients b_n in the expansion

$$\zeta(s) L(s, X_1) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \quad (\sigma > 1) \quad (9)$$

are different from zero.

$\mathcal{G}(N)$ will denote the system of numbers belonging to $\mathcal{G}$ and $\leq N$; $A(N)$ denotes the quantity of such numbers; $A(N, N_1)$ denotes the quantity of numbers $n_1 \in \mathcal{G}(N)$ whose prime factors are all $\geq N_1$; $\mathcal{G}(N, N_1)$ is the set of n_1 's; $N_{\Psi} = \exp\left(0,001 \frac{\ln D}{\Psi(D)} \ln \Psi(D)\right)$ (cf. [I], § 11); $\nu = 1/\ln D$; (M, Y, z) -circle denotes the circle $|s - z| \leq Y$ containing M poles of $f(s)$; $(\leq M, Y, z)$ -circle and $(\geq M, Y, z)$ -circle are analogous notations; $c_0, c_1, c_2, \dots$ are positive constants < 1 ; $K_0, K_1, \dots, K_j = K_j(K_{j-1}, K_{j-2}, \dots, K_0)$ are constants > 1 ; $C_1, C_2, \dots$ is another set of similar constants.

§ 7. First fundamental lemma

There exist $C_1 \geq 10$, $C_2 > C_1$, $C_3, C_4, \dots$ such that for a certain $N \in [D^{C_1}, D^{C_2}]$ we have

$$A(N, N_{\Psi}) > \frac{N}{C_3 \ln D} \exp(-C_4 \Psi(D)). \quad (10)$$

In point of fact, this theorem becomes non-trivial only for $\Psi(D) < \frac{\ln D}{2C_4}$.

It is perhaps worth mentioning that the estimate (10) for $\Psi(D) \gg \frac{\ln D}{\ln \ln D}$ immediately implies Siegel's class-number theorem for $k(\sqrt{-D})$.

We shall prove this theorem in using the lemmas of the papers [I] and [II]; the case $\Psi(D) \gg (\ln D)^{0,001}$ is rather simple and the arguments of [I] are sufficient to treat it. The case $\Psi(D) \in [c_0, (\ln D)^{0,001}]$ is treated by the complicated methods of [II].

§ 8

Lemma I. If ρ denotes the typical pole of $f(s)$, $0 < \sigma \leq 2$, then

$$\left| f(s) - \sum_{|p-s| \leq 1} \frac{1}{s-\rho} \right| \leq C_5 \ln D (|t| + 2). \quad (11)$$

Proof. As $D_1 D_2 < D^2$, this follows from [I], Lemma V, § 6.

Lemma II. Any circle with the radius $r \in \left[\frac{1}{\ln D}, 2 \right]$ and the center $1 + it$ contains no more than $C_6 r \ln D (|t| + 2)$ poles of $f(s)$, each counted times its residue.

Proof. These poles are the zeros of $L(s, X_2) \cdot L(s + \sigma_0, X_1 X_2)$ and, as $\sigma_0 \leq \frac{c_0}{\ln D}$, this follows from [1], § 5, Lemma IV.

Lemma III. The function $f(s)$ has a pole $\rho_2 = \beta_2 + it_2$, subject to following condition:

$$(1) \quad \beta_2 \geq 1 - \frac{\Psi(D)}{\ln D}.$$

$$(2) \quad |t_2| \leq \ln^8 D.$$

(3) There are no other poles in

$$\sigma > \beta_2 + \frac{1}{\ln D} = \beta_2 + \mu, \quad |t - t_2| \leq \ln^3 D.$$

Proof. $\rho_1 = \beta_1 + it_1$ is the pole with $\beta_1 = 1 - \frac{\Psi(D)}{\ln D}$. If there are other poles in $\sigma > \beta_1 + \mu$, $|t - t_1| \leq \ln^3 D$, we proceed by the «shifting process» as in [1], Lemma I, and obtain the required pole in no more than $\ln D$ steps.

§ 9

We shall now prove (10) for $\Psi(D) \in \left[(\ln D)^{0.991}, \frac{\ln D}{3} \right]$ closely following the argument of the paper [1]. We shall mark the corresponding steps and refer for the proof to the paper [1].

Lemma IV. On the line $\sigma = \sigma_2 = \beta_2 + C_7 \mu$ there exists a point $s_2 = \beta_2 + it$ with the following properties:

$$(1) \quad |\tau_2 - t_2| \leq \ln D;$$

$$(2) \quad |f(s_2)| = P \ln D \text{ with } C_8 \ln D \geq P \geq \frac{C_7}{10};$$

$$(3) \quad |f(s_3)| > \frac{P}{2} \ln D \text{ for } s_3 = s_2 + r; \quad r = \frac{1}{10^7} C_7 \mu;$$

$$(4) \quad |f(s)| \leq 2P \ln D \text{ for } s = \sigma_2 + it; \quad |t - t_2| \leq (\ln \ln D)^2 \quad \left(\mu = \frac{1}{\ln D} \right).$$

Proof. Cf. [1], §§ 2, 3, 4.

Lemma V. There exists a point $s_4 = \sigma_4 + i\tau_4$ under conditions:

$$(1) \quad \sigma_4 = \sigma_0 + \frac{\ln \Psi(D)}{C_9 \ln D};$$

$$(2) \quad |\tau_4 - \tau_2| \leq (\ln \ln D)^2;$$

$$(3) \quad |f(s_4)| > C_{10} \frac{P \ln D}{(\Psi(D))^{1/2}}.$$

Proof. Cf. [1], §§ 5, 6.

Lemma VI. For $\delta_1 = D^{-C_9}$ we have

$$\left| \sum_{p \geq N_2} \frac{a_p \ln p}{p^{s_4}} e^{-\delta_1 p} \right| > C_{11} (\ln D)^{1 - \frac{0.991}{2}}. \quad (12)$$

Proof. We obtain proof of this lemma by J. E. Littlewood's summation method with the aid of Γ -function (cf. [1], §§ 7, 8).

§ 10

We introduce now the sums $S(n) = \sum_{N_{\Psi} \leq p \leq n} \frac{a_p \ln p}{p^{i+4}}$.

Lemma VII:

$$\sum_{n \geq N_{\Psi}} |S(n)| \frac{e^{-\delta_1 n}}{n^{2-\Psi(D)/\ln D}} > C_{12} (\ln D)^{1-\frac{0.001}{2}}. \quad (13)$$

Proof. Cf. [1], §§ 7–8.

Lemma VIII. *There exists a number $N_1 \in [N_{\Psi}, \delta_1^{-2}]$ such that*

$$|S(N_1)| > N_1^{1-\alpha_D}, \quad \alpha_D = 8 \frac{\Psi(D)}{\ln D}.$$

Proof. Cf. [1], §§ 7–8.

§ 11

We choose now a positive integer ν such that $N_2 = N_1^{\nu} \in [\delta_1^{-2}, \delta_1^{-4}] = [D^{2C_9}, D^{4C_9}]$ and form then $\{S(N_1)\}^{\nu} = \sum_{n_1 \leq N_2} \xi(n_1)$, where

$$\xi(n_1) = \sum_{p_1 \dots p_{\nu} = n_1} a_{p_1} \dots a_{p_{\nu}} \ln p_1 \dots \ln p_{\nu} n_1^{-i_4 \nu}.$$

Lemma IX:

$$|\xi(n_1)| < \exp C_{13} \Psi(D).$$

Proof. $D^{4C_9} > N_2 = N_1^{\nu} \geq N_{\Psi}^{\nu}$; hence

$$\begin{aligned} \nu &\leq \frac{4C_9 \ln D}{\ln N_{\Psi}} < C_{14} \frac{\Psi(D)}{\ln \Psi(D)}, \\ |\xi(n_1)| &< \nu! 2^{\nu} \ln p_1 \dots \ln p_{\nu} \leq \nu! 2^{\nu} \left(\frac{\ln p_1 + \dots + \ln p_{\nu}}{\nu} \right)^{\nu} = \\ &= \frac{\nu! 2^{\nu}}{\nu^{\nu}} (\ln N_2)^{\nu} < C_{15}^{\nu} (\ln D)^{\nu} < C_{15}^{\nu} \exp C_{14} \frac{\Psi(D) \ln \ln D}{\ln \Psi(D)} < \\ &< C_{15}^{\nu} \exp (1000 C_{14} \Psi(D)) < \exp (C_{13} \Psi(D)), \end{aligned}$$

q. e. d.

§ 12

If $n_1 \in \mathcal{G}(N_2)$, then for at least one prime p_j in the decomposition $n_1 = p_1 \dots p_{\nu}$ we must have: $|a_{p_j}| = 1 - p_j^{-\sigma_0}$ [by (9)]. Hence

$$|\xi(n_1)| < |1 - p_j^{-\sigma_0}| \cdot \exp C_{13} \Psi(D) < |1 - N_2^{-\sigma_0}| \cdot \exp C_{13} \Psi(D).$$

We have by Lemma VIII

$$|S(N_2)| = |S(N_1)|^{\nu} > N_2^{1-\alpha_D} > N_2 \exp (-C_{16} \Psi(D)).$$

Each term of $S(N_2)$ involves only prime factors $\geq N_{\Psi}$, and hence, by Lemma IX,

$$\begin{aligned} A(N_2, N_{\Psi}) &\geq N_2 \exp (-C_{16} \Psi(D)) \exp (-C_{13} \Psi(D)) - \\ &\quad - N_2 |1 - N_2^{-\sigma_0}| \exp C_{13} \Psi(D) \geq \\ &\geq N_2 \exp (-C_{17} \Psi(D)) - N_2 |1 - N_2^{-\sigma_0}| \exp C_{13} \Psi(D). \end{aligned}$$

We may now distinguish two cases:

I. $1 - N_2^{-\sigma_0} \leq \exp(-2(C_{13} + C_{17})\Psi(D))$. In this case, $A(N_2, N_{\Psi}) \geq \frac{1}{2} N_2 \exp(-C_{17}\Psi(D))$ and so (10) is proved.

II. $1 - N_2^{-\sigma_0} > \exp(-C_{18}\Psi(D))$, where $C_{18} = 2(C_{13} + C_{17})$. Then we must have: $\sigma_0 > \frac{1}{C_{19} \ln D} \exp(-C_{20}\Psi(D))$, as $N_2 \leq \exp(4C_9 \ln D)$. Hence

$$\begin{aligned} \exp(C_{20}\Psi(D)) &> \frac{1}{C_{19}\sigma_0 \ln D} = \frac{\mu_0}{C_{19}}, \\ \Psi(D) &> \frac{1}{C_{20}} \ln \frac{\mu_0}{C_{19}} = \frac{1}{C_{20}} \ln \mu_0 - \frac{\ln C_{19}}{C_{20}} > \frac{1}{2C_{20}} \ln \mu_0; \end{aligned}$$

since

$$\Psi(D) \geq (\ln D)^{0,001} > 2 \frac{\ln C_{19}}{C_{20}}. \quad (14)$$

In view of the extremal properties of $\frac{\Psi(D)}{\ln D}$ this is the second basic theorem of § 3, with $c_3 = \frac{1}{2C_{20}}$. If it holds, then (10) is of no use, since it was introduced only to prove that theorem. Though it is true even then (it may be easily deduced from the second basic theorem), we shall not prove it.

§ 13

We proceed now to the proof of (10) for

$$\Psi(D) \in \left[\frac{c_0}{4}, (\ln D)^{0,001} \right].$$

We put $K_0 = (C_5 C_6 + 100)^{100}$, $K_1 = K_0^{0,1}$, $K_2 = K_1^2$ and proceed as in [I], § 8.

Let $\rho_2 = \beta_2 + i\eta_0$ be the pole of Lemma III ($\eta_0 = t_2$). We take $\alpha_0 = \beta_2 + K_1 \mu$ and consider the segment $\mathfrak{S}_{\alpha_0} = \mathfrak{S}(\alpha_0, |t - \eta_0| \leq \ln^3 D)$.

If $R \geq \Psi(D)$, then, by virtue of Lemma II and of the inequality $\beta_2 \geq 1 - \frac{\Psi(D)}{\ln D}$, any circle $|s - z_0| \leq R\mu$ with $z_0 \in \mathfrak{S}_{\alpha_0}$ contains no more than $10C_6 R$ poles of $f(s)$, i. e. is a $(\leq 10C_6 R, R\mu, z_0)$ -circle.

We consider now two cases:

I*. The analogous property holds also for the circles with $R \in [4K_2, \Psi(D)]$, or, more precisely, any circle $|s - z_0| \leq R\mu$ with $z_0 \in \mathfrak{S}_{\alpha_0}$; $R \in [4K_1, \Psi(D)]$ is a $(\leq K_2 R, R\mu, z_0)$ -circle.

II*. The property does not hold, and there is a $z_0 \in \mathfrak{S}_{\alpha_0}$ and an $R \in [4K_1, \Psi(D)]$ such that the circle $|s - z_0| \leq R\mu$ is a $(\geq K_2 R, R\mu, z_0)$ -circle.

§ 14

We shall at first discuss the easier case I*. In this case, we shall prove even a sharper result than (10).

Lemma IX. In case I*, we have for a certain $K_3 > 10$ and $N = D^{K_3}$: the number of primes belonging to $\mathfrak{G}(N)$ is

$$> \frac{N}{K_4 \ln D} \exp(-K_5 \Psi(D)).$$

In particular, (10) holds, since the number of primes $\leq N_{\Psi}$ is $\leq N_{\Psi} < D$.

Proof. We obtain, as in [I], § 9, proof of the Lemma VI, by J. E. Littlewood's summation method for $\delta \in (0, 1)$:

$$-\sum a_n \Delta(n) n^{-i\eta_0} \cdot e^{-\delta n} = \sum_{\rho_k \in \mathfrak{S}} m_k \Gamma(\rho_k - i\eta_0) \delta^{i\eta_0} \delta^{-\rho_k} + O(\ln^2 D). \quad (15)$$

Here ρ_k runs over the poles of $f(s)$ in the stripe $-\sigma_0 < \sigma \leq 1$, and m_k is the multiplicity of the corresponding zero of $F(s) = L(s, X_2) \cdot L(s + \sigma_0, X_1 X_2)$. Denoting here $(-\ln \delta) = x$, we can write (cf. [I], § 9)

$$\left| \sum_{\rho_k \in \mathfrak{S}} m_k \Gamma(\rho_k - i\eta_0) \delta^{i\eta_0} \cdot \delta^{-\rho_k} \right| = \delta^{-\beta_2 - \mu} \left| \sum_{k=-\infty}^{\infty} \Gamma(\rho_k - i\eta_0) \exp(-\sigma'_k + it_k) x \right|.$$

Here $\Psi_0(x) = \sum_{k=-\infty}^{\infty} \Gamma(\rho_k - i\eta_0) \exp(-\sigma'_k + it_k) x$ are the exponential series satisfying the five conditions of the Lemma II in [I], § 4, with $-\sigma'_k + it_k = \rho_k - \beta_2 - \mu$; β_2 instead of β_0 in [I], and the condition (3) is replaced by a stronger condition:

$$\beta_k \leq \beta_2 + \mu \text{ for } |t - \eta_0| \leq \ln^2 D, \quad \Delta = \ln D, \quad A_1 = C_6, \quad A_2 = K_2.$$

Hence, by this Lemma, there exist K_6, K_7, K_8 all > 10 and depending on K_2 and C_6 only such that for $X_1 = K_6 \ln D$; $X_2 = K_7 \ln D$ we have:

$$\int_{X_1}^{X_2} |\Psi_0(x)|^2 dx > \frac{\ln D}{K_8}. \quad (16)$$

Hence we obtain for a certain $\xi \in [X_1, X_2]$:

$$|\Psi_0(\xi)| > \frac{1}{K_8 K_7} = \frac{1}{K_9}.$$

Denoting $e^{\pm i} = \delta_1^{-1} = N_1 \in [D^{K_6}, D^{K_7}]$ and taking into account that $\beta_2 \geq 1 - \frac{\Psi(D)}{\ln D}$ we obtain from (15)

$$\left| \sum_p a_p \Delta(p) p^{-i\eta_0} \cdot e^{-p/N_1} \right| > N_1 \frac{\exp(-K_{10} \Psi(D))}{K_9}$$

and *a fortiori*

$$\left| \sum_p |a_p| \ln p \cdot e^{-p/N_1} \right| > \frac{N_1}{K_9} \exp(-K_{10} \Psi(D)). \quad (17)$$

Hence either (14), or the second basic theorem follow by the same reasoning, as in § 13 of the present paper. (14) follows from the second basic theorem, but is of no use if the theorem is proved.

§ 15

We shall consider now the case II of § 14. Here we proceed by the methods of [I], §§ 9–26.

Lemma X. *In case II of § 14 there exists a point $z_1 = \alpha_1 + i\eta_1$ subject to following conditions:*

(1) $\alpha_1 = \beta_2 + 2^{j_1} \cdot K_1 \mu = \beta_2 + \Delta_1 \mu$ with $j_1 \in [1, 2 \ln \Psi(D)]$ integer, $|\eta_1| \leq (\ln D)^{10}$.

(2) The circle $|s - z_1| \leq 4 \cdot 2^{j_1} \cdot K_1 u = 4\Delta_1 u$ is a $(P\Delta_1, 4\Delta_1 u, z_1)$ -circle with $P \in \left[\frac{K_2}{10}, 2C_2\Psi(D) \right]$.

(3) The segment $\mathfrak{S}[\alpha_1, |t - \eta_1| \leq \ln^2 D]$ bears only $(\leq 2P\Delta_1, 4\Delta_1 u, z)$ -circles.

(4) The segments $\mathfrak{S}[\sigma_j, |t - \eta_1| \leq \ln^2 D]$ with $\sigma_j = \alpha_1 + 2^j \Delta_1 u$; $j = 1, 2, \dots, [2 \ln \Psi(D)]$ bear only $(\leq 8K_2 \cdot 2^j \Delta_1, 4 \cdot 2^j \Delta_1 u, z)$ -circles.

Proof. Cf. [I], § 9, Lemma VIII.

§ 16

In $\sigma \geq \beta_2 + \frac{3}{2} u$, $|t - \eta_1| \leq \ln^2 D$, the function $f(s)$ is regular by construction of z_1 (cf. [I]).

We introduce now the function (cf. (7), § 5)

$$B_{\Psi}(s) = - \sum_{p \leq N_{\Psi}} \frac{a_p \ln p}{p^s} - \sum_{m=2}^{\infty} \sum_p \frac{a_p \ln p}{p^{m \cdot s}}.$$

Here p denotes a typical prime. For $\sigma > \frac{5}{9}$, we have obviously:

$$B_{\Psi}(s) = - \sum_{p \leq N_{\Psi}} \frac{a_p}{p^s} + O(1).$$

Lemma XI. In $\sigma \geq 1 - \frac{R}{\ln D}$, $10\Psi(D) \leq R \leq \frac{4}{9} \ln D$, we have

$$|B_{\Psi}(s)| < C_{21} \frac{\ln D}{\Psi(D)} \ln \Psi(D) \exp \left(0,001 \frac{R}{\Psi(D)} \ln \Psi(D) \right).$$

In particular, in $\sigma \geq 1 - \frac{10\Psi(D)}{\ln D}$

$$|B_{\Psi}(s)| \leq \frac{\ln D}{(\Psi(D))^{2/3}}.$$

Here we assume $\Psi(D) > 10$.

Proof. Cf. [I], § 11, Lemma X.

§ 17

We now introduce the functions

$$f_1(s) = f(s) - B_{\Psi}(s), \quad \varphi(s) = \frac{f_1(s)}{s - i\eta_0}.$$

As was explained in [I], § 12, we shall try to avoid the «density difficulties» of the direct study of the poles of $f(s)$ by replacing the function $f(s)$ by a new formation $Q(w) = \frac{\varphi'(w)}{\varphi(w) - Z}$, Z being a suitable constant for a given D . We consider the maxima $M_j = M_{\sigma_j}$ of $|\varphi(s)|$ on the segments $\mathfrak{S}[\sigma_j, |t - \eta_1| \leq \ln^2 D]$ for $\sigma_j = \beta_2 + 2^j \Delta_1 u$ ($j = 0, 1, 2, \dots, l$) (cf. [I], § 14).

Lemma XII. There exists an integer $j_0 \in [0, l-1]$ such that

$$M_{j_0+1} \leq \left(1 - \frac{1}{(j_0+2)^2} \right) M_{j_0}, \quad M_{j+1} \geq \left(1 - \frac{1}{(j+2)^2} \right) M_j \quad (j < j_0),$$

$$M_{j_0} > 10^{-6} P \ln D.$$

(P is the number of Lemma X).

Proof. Cf. [I], Lemma XII; §§ 14 and 15.

Let v be the point of $\mathfrak{S}[\sigma_{j_0}; |t - \tau_1| \leq \ln^2 D]$ which is the nearest to $i\tau_1$ and such that $\varphi(v) = M_{j_0}$.

Lemma XIII. *If $z \in \mathfrak{S}[\sigma_{j_0}; |t - \tau_1| \leq \ln^2 D]$, then the equation $\varphi(s) - \varphi(v) = \varphi_1(s) = 0$ has in the circle $|s - z| \leq 10^{-5} \cdot 2^{j_0} \cdot \Delta_{1, \mu}$ no more than $K_{11} \ln(j_0 + 2)$ zeros.*

Proof. Cf. [I], § 16, Lemma XIII.

Lemma XIV. *If $R \in [\Delta_1 \cdot 2^{j_0}, \Delta_1 \cdot 2^{j_0} \ln \ln D]$, then the quantity of zeros of $\varphi_1(s)$ in $|s - v_1| \leq R\mu$ does not exceed $K_{12}R$, K_{12} depending on K_1 only.*

§ 18

Lemma XV. *The quantity of zeros and poles of $\varphi_1(s)$ in any circle of the type $|s - (1 + iT)| \leq \frac{4}{10}$ does not exceed $C_{22} \ln D (|T| + 2)$.*

Proof. Cf. [I], § 17, Lemma XV.

Lemma XVI. $\varphi_1(s)$ has no zeros and poles in $\sigma > \sigma_{j_0}$, $|t - Iv_1| \leq \frac{(\ln D)^2}{2}$, and in $\sigma > 1 + \mu$.

Proof. The first assertion follows easily by definition of M_{j_0} . For the second cf. [I], § 19, Lemma XVII. ($Z = \varphi(v)$).

§ 19

We introduce now $Q(w) = \frac{\varphi'(v)}{Z - \varphi(w)}$ with $Z = \varphi(v)$ and consider the integral

$$\Phi(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \delta^{-w} \Gamma(w - i\tau_1) Q(w) dw, \quad (18)$$

where $\tau_1 = Iv$ so that $v = \sigma_{j_0} + i\tau_1$.

Then we obtain (cf. [I], § 19 and § 20, Lemma XVIII):

$$\Phi(x) = \sum_{\rho_k} \pm m_k \Gamma(\rho_k - i\tau_1) \delta^{-\rho_k} + R_L, \quad (19)$$

where ρ_k runs over the poles of $Q(w)$ between $\sigma = 1 + \mu$ and the contour L (belonging to the stripe $0,6 \leq \sigma \leq 0,65$)

$$|R_L| < \delta^{-0,65} \ln^{10} D. \quad (20)$$

We denote now $\sigma_{j_0} = \alpha_0$, $v = \alpha_0 + i\tau_1 = \rho_0$. Denoting $\rho_k - \rho_0 = -\sigma_k + it_k$, so that $-\sigma_k \leq 0$ for $|t_k| \leq (\ln D)^2$, $-\sigma_0 + it_0$, $\Gamma_j = \Gamma(\rho_j + i\tau_1)$, $-\ln \delta = x$, we obtain

$$\Phi(x) = e^{\alpha_0 x} \sum_{k=-\infty}^{\infty} \pm m_j \Gamma_j \cdot \exp(-\sigma_j + it_j)x + R_L,$$

where the sign (+) is to be taken for the zeros of $\varphi_1(s)$, the sign (−) for the poles, $-m_j \leq -1$ and $+m_j$ equals to the multiplicity of the zero ρ_j .

It must be remarked that for $|t_j| \leq \frac{\ln^2 D}{2}$ we have the sign $(-)$ only if $-\sigma_j < \frac{1}{2} \cdot 2^{j_0} \cdot \Delta_1 u = \frac{r_0 k}{2}$.

L e m m a X V I I. For $\Psi(D) > K_{13}$ there exist two constants K_{14} and K_{15} depending on $K_1, K_2, \dots, K_{12}$ only ($K_{14} > 10; K_{15} > K_{14}$) such that for $X_1 = K_{14} \ln D, X_2 = K_{15} \ln D$ we have

$$\int_{X_1}^{X_2} |S(x)|^2 dx > \frac{\ln D}{(\Psi(D))^2}, \quad (21)$$

where

$$S(x) = \sum \pm m_j \exp(-\sigma_j + it_j)x.$$

P r o o f. This is a straightforward analogon of the crucial Lemma XIX, [1]. Cf. [1], § 21, Lemmas XIX, XX.

We must remark that if $\Psi(D) < K_{13}$, then we have the case I of § 15 with K_{16} instead of K_2 and then the first fundamental lemma of § 7 is proved. Hence in what follows we assume $\Psi(D) > K_{13}$.

§ 20

We introduce now the transform (cf. [1], § 24):

$$E(y, k) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^{-w} \frac{\Gamma(w - i\tau_1)}{(w - i\eta_0)^k} dw,$$

and obtain ([1], § 26)

$$\begin{aligned} \Phi(x) = & \sum_{m=0}^{\ln D} \frac{(-1)^m}{Z^m} \cdot \sum_{n=2}^{\infty} \left\{ \frac{V_{m+1}(n) \ln n}{m+1} E(\delta n, m+1) + \right. \\ & \left. + V_{m+1}(n) \cdot E(\delta n, m+2) \right\} + R_1 \end{aligned}$$

with $|R_1| < \exp\left(-\frac{1}{2} \ln D \ln \ln D\right)$ provided $\delta^{-1} < \exp(\ln D (\ln \ln D)^{1/2})$.

Here

$$\begin{aligned} V_{m+1}(n) = & \sum_{p_1 \dots p_{m+1} = n} a_{p_1} \dots a_{p_{m+1}} \ln p_1 \dots \ln p_{m+1}, \\ V_{m+1}(n) < & 10^4 \cdot 3^m \cdot (\ln n)^{m+1} \quad (\text{cf. [1], § 23; (74)}). \end{aligned} \quad (22)$$

L e m m a X V I I I. For $M_{\Psi} = K_{16} \frac{\Psi(D)}{\ln \Psi(D)}$, $x \in [X_1, X_2]$ we have

$$\Phi(x) \leq \sum_{m=0}^{M_{\Psi}} \frac{1}{(\ln D)^{m+1}} \left| \sum_m \right| + R_2, \quad x = -\ln \delta \quad (23)$$

$$|R_2| < 2 \exp\left(-\frac{1}{2} \ln D \cdot \ln \ln D\right).$$

Here

$$\sum_m = \sum_{n=2}^{\infty} \left\{ \frac{V_{m+1}(n) \ln n}{m+1} E(\delta n, m+1) + V_{m+1}(n) \cdot E(\delta n, m+2) \right\}.$$

Proof. Cf. [I], § 25, Lemma XXII.

§ 21

From (19) and (21) we conclude that there exists a $\xi \in [X_1, X_2]$ such that

$$|\Phi(\xi)| > \frac{1}{K_{17}(\Psi(D))} e^{\xi}. \quad (24)$$

And hence, in virtue of (22), there exists an $m_0 \leq M_{\Psi}$ such that

$$\frac{\sum_{m_0} |}{(\ln D)^{m_0}} > \frac{1}{K_{18}(\Psi(D))^3} N_1, \quad N_1 = e^{\xi} = \delta^{-1}. \quad (25)$$

We denote now $A(n) = \prod_{p|n} a_p$. In virtue of (22) and the estimates for $E(y, m_0+1)$ and $E(y, m_0+2)$ ([I], § 24, Lemma XXI), we get from (25):

$$\sum_{N_1^{0.99} \leq n \leq DN_1} \frac{V_{m_0+1}(n)}{(\ln D)^{m_0+1}} \left[\frac{\ln n}{m_0+1} \left| E\left(\frac{n}{N_2}, m_0+1\right) \right| + \left| E\left(\frac{n}{N_1}, m_0+2\right) \right| \right] > \frac{N_1}{2K_{18}(\Psi(D))^3}. \quad (26)$$

We divide the segment $[N_1^{0.99}, N_1]$ into partial segments of the type $\left[\frac{N_1}{2^{r+1}}, \frac{N_1}{2^r}\right]$; the extreme left segment may be not representable in this form, but it may be omitted (cf. [I], § 28); the segment $[N_1, N_1 D]$ is divided into the segments $[N_1 \cdot 2^r, N_1 \cdot 2^{r+1}]$ with an analogous proviso.

We have then, by [I], § 28:

$$\left| E\left(\frac{n}{N_2}, m_0+1\right) \right| + \left| E\left(\frac{n}{N_1}, m_0+2\right) \right| \leq 2^{\frac{r+1}{2}} \exp(K_{19} \Psi(D)) \quad (27)$$

in the segment $\left[\frac{N_1}{2^{r+1}}, \frac{N_1}{2^r}\right] \subset [N_1^{0.99}, N_1]$

$$\left| E\left(\frac{n}{N_1}, m_0+1\right) \right| + \left| E\left(\frac{n}{N_1}, m_0+2\right) \right| \leq \exp\left(-\frac{2^r}{4}\right) \exp(K_{19} \Psi(D)) \quad (28)$$

in the segment $[N_1 \cdot 2^r, N_1 \cdot 2^{r+1}] \subset [N_1, N_1 D]$.

§ 22

Lemma XIX. For a suitable r_0 we have either

$$\sum_{N_1/2^{r_0+1} \leq n_1 \leq N_1/2^{r_0}} A(n_1) \ln n_1 > \frac{N_1}{2^{r_0}} \exp(-K_{21} \Psi(D)),$$

or

$$\sum_{N_1 \cdot 2^{r_0} \leq n_1 \leq N \cdot 2^{r_0+1}} A(n_1) \ln n_1 > N_1 \cdot 2^{r_0} \exp(-K_{21} \Psi(D));$$

the summation runs over n_1 's with $V_{m+1}(n_1) \neq 0$.

Proof. Suppose that K_{21} being not fixed, these estimations do not hold. Taking into account the estimates (22), (26), (27) and (28), we get, as $N_1 < e^{X_2}$:

$$\begin{aligned} \sum_{N_1^{0.99} \leq n \leq D N_1} \frac{V_{m_0+1}(n)}{(\ln D)^{m_0+1}} \left[\frac{\ln n}{m_0+1} \left| E\left(\frac{n}{N_1}, m_0+1\right) \right| + \left| E\left(\frac{n}{N_1}, m_0+2\right) \right| \right] < \\ < K_{15}^{m_0+1} \cdot \exp(K_{15} \Psi(D)) \cdot \left\{ \sum_{r=0}^{\infty} \frac{2^{\frac{r+1}{2}}}{2^r} + \right. \\ \left. + \sum_{r=0}^{\infty} 2^r \cdot \exp(-2^r/4) \right\} \exp(-K_{21} \Psi(D)) N_1, \end{aligned} \quad (29)$$

$$\begin{aligned} K_{15}^{m_0+1} < K_{15}^{2M_\Psi} < \exp(2M_\Psi \ln K_{15}) = \exp\left(2K_{15} \ln K_{15} \frac{\Psi(D)}{\ln \Psi(D)}\right) < \\ < \exp(K_{20} \Psi(D)). \end{aligned}$$

Hence, if $K_{21} \geq 100(K_{15} + K_{20})$, we obtain that the left hand side of (26) is $N_1 \exp\left(-\frac{K_{20}}{2} \Psi(D)\right)$. If $K_{20} \geq 6K_{15}$ this is obviously impossible. We may now put $K_{21} = 100(K_{15} + K_{19} + K_{20})$, and the lemma is proved.

§ 23

By Lemma XIX, there exists a number $N_2 \in [N_1^{0.99}, N_1 D]$ such that

$$\sum_{N_2/2 \leq n_1 \leq N_2} A(n_1) \ln n_1 > N_2 \exp(-K_{20} \Psi(D)), \quad (30)$$

where n_1 's run over the numbers with $V_{m_0+1}(n_1) \neq 0$. Each number n_1 consists of m_0+1 prime factors $> N_\Psi$. Hence we obtain

$$\sum_{N_2/2 \leq n_1 \leq N_2} A(n_1) > \frac{N_2}{K_{15} \ln D} \exp(-K_{20} \Psi(D)). \quad (31)$$

Lemma XX. We have: either

$$A(N_2, N_\Psi) > \frac{N_2}{K_{22} \ln D} \exp(-K_{23} \Psi(D)) \quad (32)$$

or the second basic theorem is true.

Proof. By the same argument as that of § 13. Hence, either the second basic theorem, or the first fundamental lemma of § 7, is true. We have now to prove that this theorem follows from the fundamental lemma, i. e. from (10). To this end, we shall prove the «second fundamental lemma».

§ 24

Returning to the relation (9), we put:

$$\sum_{n \leq D^{0.52}} \frac{b_n}{n} = W. \quad (33)$$

Second fundamental lemma:

$$A(N_2, N_\Psi) < C_{29} L(1, X_1) \frac{N_2}{W} (\Psi(D))^2 \text{ for } N \geq D^s. \quad (34)$$

Proof. For proof we shall apply Viggo Brun's sieve method. To this end, we need some further lemmas. We have by (9) in $\sigma > 1$

$$\zeta(s) \cdot L(s, X) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \quad b_n \neq 0 \text{ if and only if } n \in \mathcal{G}.$$

The left hand side can be decomposed in $\sigma > 1$ into the infinite product

$$\zeta(s) L(s, X_1) = \prod_p P(p), \quad P(p) = \frac{1}{1-p^{-2s}} \text{ for } X_1(p) = -1;$$

$$\frac{1}{1-p^{-s}} \text{ for } X_1(p) = 0, \quad \frac{1}{(1-p^{-s})^2} \text{ for } X_1(p) = +1.$$

We introduce now the function

$$\Phi(s) = \prod_{p \in \mathcal{G}} \frac{1}{1-2p^{-s}} = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \quad (\sigma > 1). \quad (35)$$

We have

$$\Phi(s) = \zeta(s) L(s, X) \cdot \prod_{p/D_1} \left(1 - \frac{1}{p^s}\right) R_{D_1}(s), \quad (36)$$

where $R_{D_1}(s)$ is a Dirichlet's series which is absolutely convergent in $\sigma > \frac{1}{2}$ and so is $O(1)$ in $\sigma \geq 0,51$.

§ 25

Lemma XXI. For $N \geq 1$ we have

$$C(N) = \sum_{n \leq N} c_n = N R_{D_1}(1) L(1, X_1) \cdot P_{D_1} + O(N^{0,52} \cdot D^{1/2}), \quad (37)$$

$$P_{D_1} = \prod_{p/D_1} \left(1 - \frac{1}{p}\right).$$

Proof. Using (36), we obtain with $a = 1 + \frac{1}{\ln N}$:

$$C(N) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{a-iT}^{a+iT} \frac{N^s}{s} \Phi(s) ds + O(1).$$

It is well known (cf. Page, [7]), that

$$\left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{N^s}{s} n^{-s} ds - 1 \right| \leq \frac{1}{T} N^a \frac{n^{-a}}{\left| \ln \frac{N}{n} \right|} \text{ for } n < N;$$

$$\left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{N^s}{s} n^{-s} ds \right| \leq \frac{1}{T} N^a \frac{n^{-a}}{\left| \ln \frac{N}{n} \right|} \text{ for } n > N.$$

Taking into account these estimates, and putting $T=N$, we move the vertical segment to $\delta=0,51$, passing the pole $s=1$ with the residue $L(1, X_1) \cdot R_{D_1}(1) \cdot P_{D_1}$. The remainder term is then easily proved to be of order $N^{0,52} \cdot D^{1/2}$ (cf. Page, [7]).

§ 26

If $c_{n_1} \neq 0$ and $n_1 = q_{\alpha_1} \dots q_{\alpha_r} \cdot n'_1$ being primes, then $c_{n_1} = 2^r c_{n'_1}$. Hence the quantity $\sum_{n_1 \leq N} c_{n_1}$, where n_1 runs over all the numbers such that $n_1 = q_{\alpha_1} \dots q_{\alpha_r} n'_1$, equals to $2^r \sum_{n_1 \leq N} c_{n_1}$, where $N' = \frac{N}{q_{\alpha_1} \dots q_{\alpha_r}}$ and n runs over all numbers $\leq N'$. This remarks and the estimate (37) enable us to apply the sieve method.

We shall use the adaptation of Viggo Brun's method exposed in Schnirelmann's celebrated work «Über additive Eigenschaften von Zahlen» (Math. Annalen, 107, 1933). We shall quote it after the russian translation [10].

Let $q_1, q_2, \dots, q_r$ be all primes $\leq N_{\Psi}; \in \mathcal{G}$. We put $C(N; q_1, q_2, \dots, q_r) = \sum_{n_1 \leq N} c_{n_1}$ the sum being extended over all n'_1 's which are not divisible by $q_1, q_2, \dots, q_r$.

The symbolic formula (63) (Schnirelmann, [10]), being adapted to our $C(N, q_1, \dots, q_r)$ holds, in virtue of the preceding remarks:

$$\begin{aligned} C(N, q_1, q_2, \dots, q_r) &\leq C(N) - 2 \sum_{\alpha=1}^r C\left(\frac{N}{q_{\alpha}}\right) + 4 \sum_{\alpha=1}^r \sum_{\beta < \alpha} C\left(\frac{N}{q_{\alpha} q_{\beta}}\right) - \\ &- 8 \sum_{\alpha=1}^r \sum_{\beta < \alpha} \sum_{\lambda=1}^{\min(x-1, r_{n-1})} C\left(\frac{N}{q_{\alpha} \dots q_{\lambda} q_{\lambda}}\right) + 2^{2n} \sum_{\alpha=1}^r \dots \\ &\dots \sum_{\lambda=1}^{\min(x-1, r_{n-1})} \sum_{\mu < \lambda} C\left(\frac{N}{q_{\alpha} q_{\beta} \dots q_{\lambda} q_{\mu}}\right). \end{aligned}$$

Here $1 \leq r_n < r_{n-1} < \dots < r_2 < r_1 < r$ is any fixed sequence of indices.

We have now, by (37)

$$C(N_1) = N_1 \cdot R_{D_1}(1) \cdot L(1, X_1) \cdot P_{D_1} + \theta \cdot N_1^{0,52} D^{1/2}.$$

Hence

$$\begin{aligned} C(N; q_1, \dots, q_r) &\leq L(1, X_1) R_{D_1} P_{D_1} \left\{ 1 - 2 \sum_{\alpha=1}^r \frac{1}{q_{\alpha}} + 4 \sum_{\alpha=1}^r \sum_{\beta < \alpha} \frac{1}{q_{\alpha} q_{\beta}} - \dots + \right. \\ &\left. + 2^{2n} \sum_{\alpha=1}^r \sum_{\beta < \alpha} \dots \sum_{\lambda=1}^{\min(x-1, r_{n-1})} \frac{1}{q_{\alpha} q_{\beta} \dots q_{\lambda} q_{\mu}} \right\} + R; \end{aligned}$$

$$R \leq (2r+1)^2 (2r_1+1)^2 \dots (2r_{n-1}+1)^2 N_1^{0,52} D^{1/2} \quad (\text{cf [10], (68)}).$$

We consider now the elementary symmetric functions $S_m^{(1)}, S_m^{(2)}, \dots$ of the quantities $\frac{2}{q_{r_m+1}}, \frac{2}{q_{r_m+2}}, \dots, \frac{2}{q_{r_m-1}}$. It is proved in [10], that

if $0 < S_m^{(1)} < \ln h_0$ with $h_0 = 1,29$; $m = 1, 2, \dots, n$, then the sum E in the figured brackets $\{ \}$

$$E < 2,1 \prod_{v=1}^r \left(1 - \frac{2}{q_v}\right).$$

We fix now the system of indices $r_1, r_2, \dots, r_n \geq 1$ by the conditions, that r_1 is the least number for which

$$0 < \sigma_2 = \sum_{v=r_2+1}^{r_1} \frac{1}{q_v} < \ln h_0, \quad \prod_1 = \prod_{v=r_1+1}^r \left(1 - \frac{2}{q_v}\right) > \frac{1}{h_0^2};$$

r_2 is the least number for which

$$0 < \sigma_2 = \sum_{v=r_2+1}^{r_1} \frac{1}{q_v} < \ln h_0, \quad \prod_2 = \prod_{v=r_2+1}^{r_1} \left(1 - \frac{2}{q_v}\right) > \frac{1}{h_0^2},$$

r_k is the least integer for which

$$0 < \sigma_k = \sum_{v=r_{k+1}}^{r_{k-1}} \frac{1}{q_v} < \ln h_0, \quad \prod_k = \prod_{v=r_{k+1}}^{r_{k-1}} \left(1 - \frac{2}{q_v}\right) > \frac{1}{h_0^2}, \dots$$

r_n is the least integer, for which

$$0 < \sigma_n = \prod_{v=r_{n+1}}^{r_{n-1}} \frac{1}{q_v} < \ln h_0, \quad \prod_n = \prod_{v=r_{n+1}}^{r_{n-1}} \left(1 - \frac{2}{q_v}\right) > \frac{1}{h_0^2}.$$

As the numbers q_j form a part of all primes $\leq N_{\mathfrak{F}}$, we have for $q_{r_k} > C_{23}$, $h = \frac{89}{69} < 1,29 = h_0$:

$$q_{r_1} \leq q_r^{1/h}, \quad q_{r_2} \leq q_r^{1/h^2}, \dots, \quad q_{r_k} \leq q_r^{1/h^k},$$

and hence by [10], (84) and (99):

$$R \leq C_{24} q_r^{\frac{2h}{h-1}} N^{0,52} D^{1/2} < C_{24} q_r^{8,6} N^{0,52} D_1^{1/2},$$

$$E < 2,1 \prod_{v=1}^r \left(1 - \frac{2}{q_v}\right).$$

As $q_r \leq N_{\mathfrak{F}} < D_1^{0,001}$ we obtain for $N \geq D_1^8$.

$$\begin{aligned} C(N; q_1, q_2, \dots, q_r) &\leq C_{26} L(1, X_1) P_{D_1} N \cdot \prod_{v=1}^r \left(1 - \frac{2}{q_v}\right) + \\ &+ \theta C_{24} N^{0,53} D^{1/2} < 2C_{25} L(1, X_1) P_{D_1} N \cdot \prod_{v=1}^r \left(1 - \frac{2}{q_v}\right) \end{aligned} \quad (38)$$

as $L(1, X_1) > \frac{c_3}{\sqrt{D_1 \ln D_1}}$ by (4). Here $P_{D_1} = \prod_{q|D_1} \left(1 - \frac{1}{q}\right)$.

We have now, by (33) and (9):

$$\begin{aligned} \sum_{\substack{n \leq D^{0,52} \\ n \in D_1^{0,52}}} \frac{b_n}{n} &= W, \quad W \leq \prod_{v=1}^r \left(1 + \frac{2}{q_v}\right) \cdot \prod_{q/D_1} \left(1 + \frac{1}{q}\right) \\ &\leq \prod_{N_{\Psi} \leq p \leq D_1^{0,52}} \left(1 + \frac{2}{p}\right) \cdot \prod_{p \leq D_1^{0,52}} \left(1 + \frac{1}{p^2}\right) \leq \\ &\leq C_{26} \leq \frac{1}{\prod_{v=1}^r \left(1 - \frac{2}{q_v}\right)} P_{D_1}^{-1} \cdot \prod_{N_{\Psi} \leq p \leq D^{0,52}} \left(1 + \frac{2}{p}\right). \end{aligned}$$

Now, $N_{\Psi} = \exp\left(0,001 \frac{\ln D}{\Psi D} \ln \Psi(D)\right)$ and so [10]

$$\prod_{N_{\Psi} \leq p \leq D^{0,52}} \left(1 + \frac{2}{p}\right) \leq C_{26} \frac{\ln^2 D_1}{\ln^2 N_{\Psi}} < C_{27} (\Psi(D))^2.$$

Hence $P_{D_1} \prod_{v=1}^r \left(1 - \frac{2}{q_v}\right) \leq C_{28} \frac{(\Psi(D))^2}{W}$ and hence for $N \geq D_1^3$

$$C(N, q_1, \dots, q_r) \leq C_{28} \frac{N}{W} L(1, X_1) (\Psi(D))^2. \quad (39)$$

This enables us to prove (34). Namely if $A_1(N, N_{\Psi})$ denotes the quantity of all quadretfrei numbers belonging to $\mathcal{G}(N, N_{\Psi})$, then we have obviously

$$A_1(N, N_{\Psi}) \leq C_{28} \frac{N}{W} L(1, X_1) (\Psi(D))^2$$

by (39) for $N \geq D^5 > D_1^3$.

Hence, for $N \geq D^{10}$:

$$\begin{aligned} A(N, N_{\Psi}) &\leq C_{28} \sum_{q \leq D^5} \frac{N}{W q^2} L(1, X_1) (\Psi(D))^2 + \\ &+ O(\sqrt{N}) < C_{29} \frac{N}{W} L(1, X_1) (\Psi(D))^2. \end{aligned} \quad (40)$$

This is (34).

§ 27

Lemma XXII:

$$L(1, X_1) > \frac{W \exp(-K_{24} \Psi(D))}{K_{25} \ln D}.$$

Proof. By the fundamental lemmas (10) and (34) we have for a certain $N_1 \in [D^{K_{14}}, D^{K_{15}}]$, $K_{14} \geq 10$:

$$A(N_1, N_{\Psi}) \leq \frac{N_1}{K_{23} \ln D} \exp(-K_{24} \Psi(D)), \quad (41)$$

$$L(1, X_1) \leq W \frac{1}{C_{28}} \frac{A(N_1, N_{\Psi})}{N_1} \leq \frac{1}{(\Psi(D))^2}, \quad (42)$$

(41) and (42) lead immediately to (40).

§ 28. Proof of the second basic theorem

We return to (9). The series $L(s, X_1)$ has the zero $s = \beta_0$. We apply J. E. Littlewood's summation formula to (9) in the point $s = \beta_0$:

For $\delta = D_1^{-0,51}$ we have

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \delta^{\beta_0-w} \cdot \Gamma(w-\beta_0) \zeta(w) L(w, X_1) dw = \sum_{n=1}^{\infty} \frac{b_n}{n^{\beta_0}} e^{-\delta n}.$$

Moving the contour to $\sigma = \frac{1}{2}$, we pass through the only pole $w = 1$, since the simple pole $w = \beta_0$ of $\Gamma(w - \beta_0)$ is cancelled by the zero β_0 of $L(w, X_1)$. Hence we obtain

$$\sum_{n=1}^{\infty} \frac{b_n}{n^{\beta_0}} e^{-\delta n} = \delta^{\beta_0-1} \Gamma(1-\beta_0) L(1, X_1) + o(D_1^{-0,005}). \quad (43)$$

As $\beta_0 \geq 1 - \frac{c_0}{\ln D}$, $\delta = D_1^{-0,51}$, we obtain

$$\sum_{n=1}^{\infty} \frac{b_n}{n^{\beta_0}} e^{-\delta n} + o(D_1^{-0,005}) < C_{30} \sum_{n \leq D^{0,52}} \frac{b_n}{n} = C_{30} W.$$

Hence $\frac{1}{\Gamma(1-\beta_0)} \geq L(1, X_1) \delta^{\beta_0-1} \cdot \frac{1}{C_{30} W}$, $\delta^{\beta_0-1} \geq 1$; in virtue of (40)

$$\frac{1}{\Gamma(1-\beta_0)} > \frac{W \exp(-K_{24} \Psi(D))}{K_{25} \ln D} \cdot \frac{1}{C_{30} W} > \frac{\exp(-K_{24} \Psi(D))}{K_{26} \ln D}. \quad (44)$$

Now

$$\frac{1}{\Gamma(1-\beta_0)} < 2(1-\beta_0) = 2\sigma_0, \quad \sigma_0 \ln D = \frac{1}{\mu_0}.$$

Hence, by (44)

$$\frac{2}{\mu_0} > \frac{\exp(-K_{24} \Psi(D))}{K_{26}}, \quad \exp(K_{24} \Psi(D)) > \frac{\mu_0}{2K_{26}}, \quad \Psi(D) > \frac{1}{K_{24}} \ln \frac{\mu_0}{2K_{26}}. \quad (45)$$

In virtue of the extremal properties of $\Psi(D)/\ln(D)$, this is the second basic theorem of § 3 with

$$c_3 = \frac{1}{K_{24}}, \quad c_4 = \frac{1}{2K_{26}}.$$

§ 29. The weak asymptotic law

Theorem. If

$$S(N, l) = \sum_{\substack{n=1, \dots, \infty \\ n \equiv l \pmod{D}}} \Lambda(n) e^{-n/N}, \quad x = \ln N, \quad M > B_0 \ln D,$$

(B_0 a positive absolute constant), then

$$\int_M^{2M} dx \int_x^{2x} \frac{S(N, l)}{N} dx = \frac{1}{\varphi(D)} \left\{ \int_M^{2M} dx \int_x^{2x} (1 - X_1(l) \Gamma(\beta_0) e^{-\sigma_0 x}) dx \right\} \cdot \left\{ 1 + 2\theta(c_4 \mu_0)^{-\frac{M}{\ln D}} + \frac{\theta}{\ln D} \right\}. \quad (46)$$

Proof. We follow closely the proof of (98) ([I], § 30). We divide the stripe $\frac{3}{4} \leq \sigma \leq 1$ into the stripes $\mathfrak{B}_k$:

$$1 - \frac{2^{k+1} \ln(c_4 \mu_0)}{\ln D} \leq \sigma \leq 1 - \frac{2^k \ln c_4 \mu_0}{\ln D} \quad (n = 0, 1, \dots, b).$$

We cut out of each $\mathfrak{B}_k$ the rectangle

$$|t| \leq \min(\ln(c_4 \mu_0))^{100} \cdot 2^{100(k+1)}, \ln^3 D$$

and denote it by R_{k0} .

We cut out of $\mathfrak{B}_k$ ($k \geq 1$) a sequence of pairs of symmetric rectangles $R_{k0}!$

$$2^{100k} (\ln c_4 \mu_0)^{100} \leq |t| \leq 2^{200k} (\ln c_4 \mu_0)^{200},$$

$$2^{200k} (\ln c_4 \mu_0)^{200} \leq |t| \leq 2^{300k} (\ln c_4 \mu_0)^{300}, \dots, 2^{100kg_k} \leq |t| \leq \ln^2 D,$$

and denote them by $R_{k0}, R_{k1}, \dots, R_{kg_k}$. Let Q_{kn} be the quantity of L -series with zeros in R_{kn} . Then, by definition of $Q\left(\frac{\Psi(D)}{\ln D}\right)$ we shall have

$$Q_{kn} \leq Q\left(\frac{(2^k \ln c_4 \mu_0)^n}{\ln D}\right) \leq \exp(B(2^k \ln c_4 \mu_0)^{n+1})$$

by the first basic theorem; we get now, with notations analogous to those of [I], § 30

$$\frac{\Psi(D)}{N} S(N, l) = 1 + X_1(l) \Gamma(\beta_0) \exp(-\sigma_0 x) +$$

$$+ \sum_{h=1}^b \frac{Z_h}{N} + \frac{\theta}{\ln D} \quad (\text{cf. (101), [I]}),$$

$$\left| \frac{Z_h}{N} \right| \leq C_{31} \ln D \cdot \exp\left(-x c_3 \frac{2^{h-1} \ln(c_4 \mu_0)}{\ln D}\right).$$

$$\cdot \sum_{n=0}^{g_k} \exp(B \cdot (2^k \ln c_4 \mu_0)^{n+1}) \cdot \exp(-2^{100kn} (\ln(c_4 \mu_0))^n) \leq$$

$$\leq C_{31} \ln D \cdot \exp(B \cdot 2^k \ln(c_4 \mu_0)) \cdot \exp\left(-x c_3 \frac{2^{k-1} \ln(c_4 \mu_0)}{\ln D}\right).$$

This is $< \frac{1}{2^k} \ln D \cdot \exp(-B_1 \ln(c_4 \mu_0) 2^{k-1})$ for $x > B_2 \ln D$. For

$2^{k-1} \ln \geq \ln(c_4 \mu_0) \geq \ln \ln D$, $B_1 > 6$ this is $< \frac{1}{2^k} \frac{(c_4 \mu_0)^{-2}}{\ln^2 D}$. Hence, if k_0 is the least number for which $2^{k_0-1} \ln(c_4 \mu_0) \geq \ln \ln D$, then we have (cf. [I], (102)):

$$\frac{\varphi(D)}{N} S(N, l) = 1 + X_1(l) \Gamma(\beta_0) \exp(-\sigma_0 x) + \sum_{k=1}^{h_0} \frac{Z_k}{N} + \frac{2\theta}{\ln D},$$

$$\left| \int_M^{2M} dx \int_x^{2x} \frac{Z_k}{N} dx \right| < \exp \left(-2^{k-1} c_3 \frac{M \ln(c_4 \mu_0)}{\ln D} \right).$$

$$\begin{aligned} & \cdot \sum_{k=1}^{q_k} \left\{ \exp(B \cdot 2^{k(n+1)}) \exp(-2^{100kn}) \cdot \sum_{m=1}^{\infty} \frac{C_{33} 2^m \ln^3 D}{(\ln c_4 \mu_0)^2 2^{2m}} < \right. \\ & \left. < \frac{\ln^2 D}{2^{k+1}} \exp \left(-c_5 \frac{M \ln(c_4 \mu_0)}{\ln D} \right) \quad (M > B_3 \ln D). \right. \end{aligned}$$

Hence

$$\begin{aligned} \varphi(D) \int_M^{2M} dx \int_x^{2x} \frac{S(N, l)}{N} dx &= \int_M^{2M} dx \int_x^{2x} (1 - X_1(l) \Gamma(\beta_0) e^{-\sigma_0 x}) dx + \\ &+ M^2 \left(\theta \cdot \exp \left(-c_5 \frac{M \ln(c_4 \mu_0)}{\ln D} \right) + \theta \cdot \frac{(c_4 \mu_0)^{-2}}{\ln^2 D} \right). \end{aligned}$$

We have now (cf. (98), [1])

$$\int_M^{2M} dx \int_x^{2x} (1 - \Gamma(\beta_0) e^{-\sigma_0 x}) dx = \frac{3M^2}{2} - \Gamma(\beta_0) \cdot \frac{P(e^{-\sigma_0 M})}{\sigma_0^2},$$

$$P(y) = \frac{y(y-1)^2(y+2)}{2}.$$

For $\sigma_0 M \geq 0,1$ this is $> 0,001 M^2$. For $\sigma_0 M < 0,1$ this is $> \frac{3M^2}{2}$.

$\cdot 0,001 \sigma_0 M > 0,001 \frac{M^2}{\mu_0}$. Hence, for $M > B_0 \ln D$ we can write

$$\begin{aligned} \int_M^{2M} dx \int_x^{2x} \frac{S(N, l)}{N} dx &= \frac{1}{\varphi(D)} \left\{ \int_M^{2M} dx \int_x^{2x} (1 - X_1(l) \Gamma(\beta_0) e^{-\sigma_0 x}) dx \right\} \cdot \\ &\cdot \left\{ 1 + 2\theta (c_4 \mu_0)^{-c_5 \frac{M}{\ln D}} + \frac{\theta}{D} \right\}. \end{aligned}$$

This is what we set out to prove; (2) follows immediately.

In the section III of the present paper I shall give a more precise asymptotic law.

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О наименьшем простом числе в арифметической прогрессии II. «Эффект Deuring — Heilbronn'a»

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(Резюме)

В этой статье дается теорема, которая вместе с разделом «I. Основная теорема» первой части завершает доказательство неравенств

$$\lim_{D \rightarrow \infty} \frac{\ln p_{\min}(l, D)}{\ln D} < \infty, \quad (1)$$

$$p_{\min}(l, D) < D^{c_0}, \quad (2)$$

где $p_{\min}(l, D)$ — наименьшее простое число в прогрессии $Dx + l$; $(l, D) = 1$, $l \in [1, D-1]$.

Эта теорема («эффект Deuring — Heilbronn'a») относится к «исключительному характеру» $X(n) \pmod{D}$, т. е. такому, что ряд $\sum_{n=1}^{\infty} X(n) n^{-s}$ имеет реальный нуль $\beta_0 > 1 - \frac{c_0}{\ln D}$, и гласит:

если $\sigma_0 = 1 - \beta_0$, $\mu_0 = \frac{1}{\sigma_0 \ln D} > 1$, то прямоугольник

$$1 \geq \sigma \geq 1 - c_3 \frac{\ln(c_4 \mu_0)}{\ln D}, \quad |t| \leq \ln^3 D$$

не содержит нулей ни одного $L(s, \chi)$ по всем модулям $\leq D^2$ (в частности, $\pmod{D}$), кроме, разумеется, одного β_0 . Далее доказываются «слабые асимптотические закономерности». Пусть

$$S(N, l) = \sum_{\substack{n=1, \dots, \infty \\ n \equiv l \pmod{D}}} \Lambda(n) e^{-n/N}, \quad x = \ln N.$$

Если модуль D не имеет исключительного характера $X(n)$, то при $M > C_1 \ln D$

$$\int_M^{2M} dx \int_x^{2x} \frac{S(N, l)}{N} dx = \frac{1}{\varphi(D)} \frac{3M^2}{2} \left(1 + 2\theta \cdot \exp\left(\frac{-c_1 M}{\ln D}\right) \right).$$

Если имеется исключительный $X(n)$ с соответствующим нулем β_0 , $\sigma_0 = 1 - \beta_0$, $\mu_0 = \frac{1}{\sigma_0 \ln D}$, то

$$\int_M^{2M} dx \int_x^{2x} \frac{S(N, l)}{N} dx = \frac{1}{\varphi(D)} \int_M^{2M} dx \int_x^{2x} (1 - X(l) \Gamma(\beta_0) e^{-\sigma_0 x}) dx \cdot \\ \cdot \left\{ 1 + 2\theta (c_4 \mu_0)^{-c_5 \frac{M}{\ln D}} + \frac{\theta}{\ln^2 D} \right\}.$$