

BUBBLING OUT OF EINSTEIN MANIFOLDS

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In [1], [8], and [4] the following compactness theorem of the space of Einstein metrics is obtained in the spirit of Gromov theory.

THEOREM A. *Let (X_i, g_i) be a sequence of n -dimensional ($n \geq 4$) smooth manifolds and Einstein metrics on them with uniformly bounded Einstein constants $\{e_i\}$ satisfying*

$$\text{diam}(X_i, g_i) \leq D, \quad \text{vol}(X_i, g_i) \geq V \quad \text{and} \quad \int_{X_i} |R_{g_i}|^{n/2} dV_i \leq R$$

for some positive constants D, V and R , where we denote the curvature tensor of a metric g by R_g . Then there exist a subsequence $\{j\} \subset \{i\}$ and a compact Einstein orbifold (X_∞, g_∞) with a finite set of singular points $S = \{x_1, x_2, \dots, x_s\} \subset X_\infty$ (possibly empty) for which the following statements hold:

- (1) (X_j, g_j) converges to (X_∞, g_∞) in the Hausdorff distance.
- (2) There exists an into diffeomorphism $F_j: X_\infty \setminus S \rightarrow X_j$ for each j such that $F_j^* g_j$ converges to g_∞ in the C^∞ -topology on $X_\infty \setminus S$.
- (3) For every $x_a \in S$ ($a = 1, 2, \dots, s$) and j , there exist $x_{a,j} \in X_j$ and a positive number r_j such that
 - (3.a) $B(x_{a,j}, \delta)$ converges to $B(x_a, \delta)$ in the Hausdorff distance for all $\delta > 0$.
 - (3.b) $\lim_{j \rightarrow \infty} r_j = 0$.
 - (3.c) $((X_j, r_j^{-2} g_j), x_{a,j})$ converges to $((M_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance, where (M_a, h_a) is a complete, non-compact, Ricci-flat, non-flat n -manifold which is ALE, of order $n-1$ in general, of order n if (M_a, h_a) is Kähler or $n=4$.
 - (3.d) There exists an into diffeomorphism $G_j: M_a \rightarrow X_j$ such that $G_j^*(r_j^{-2} g_j)$ converges to h_a in the C^∞ -topology on M_a .
- (4) It holds that

$$\lim_{j \rightarrow \infty} \int_{X_j} |R_{g_j}|^{n/2} dV_j \geq \int_{X_\infty} |R_{g_\infty}|^{n/2} dV_\infty + \sum_a \int_{M_a} |R_{h_a}|^{n/2} dV_{h_a}.$$

Moreover if (X_i, g_i) are Kähler, then (X_∞, g_∞) and (M_a, h_a) are also Kähler.

Here we call a smooth n -dimensional complete Riemannian orbifold (X, g) asymptotically locally Euclidean (ALE, for short) of order $\tau > 0$, if there exists a compact subset $K \subset X$ such that $X \setminus K$ has coordinates at infinity; namely there are $R > 0, 0 < \alpha < 1$, a finite subgroup $\Gamma \subset O(n)$ acting freely on $\mathbb{R}^n \setminus B(0; R)$, and a C^∞ -diffeomorphism

$\mathcal{Z}: X \setminus K \rightarrow (R^n \setminus B(0; R))/\Gamma$ such that $\varphi = \mathcal{Z}^{-1} \circ \text{proj}$ satisfies (where proj is the natural projection of R^n to R^n/Γ)

$$(\varphi^*g)_{ij}(z) = \delta_{ij} + O(|z|^{-\tau}), \quad \partial_k(\varphi^*g)_{ij}(z) = O(|z|^{-\tau-1}),$$

$$\frac{|\partial_k(\varphi^*g)_{ij}(z) - \partial_k(\varphi^*g)_{ij}(w)|}{|z-w|^\alpha} = O(\min\{|z|, |w|\}^{-\tau-1-\alpha}) \quad \text{for } z, w \in R^n \setminus B(0; R).$$

(For simplicity we assumed that (X, g) has only one end. In our Ricci-flat case this assumption is satisfied.)

Kronheimer classified all ALE hyper-Kähler surfaces of order 4 in his thesis [6], and called such manifolds ALE gravitational instantons. In particular, he proved the following:

THEOREM B. *An ALE gravitational instanton is diffeomorphic to a minimal resolution of C^2/Γ , where Γ is a finite subgroup of $SU(2)$.*

We remark that a simply connected Ricci-flat Kähler surface is hyper-Kähler. Thus in the case of Einstein-Kähler surfaces we have rather good understanding of the nature of degeneration. Only missing point is the knowledge on the neck $B(x_{a,j}; \delta) \setminus B(x_{a,j}; r_j)$, i.e., how an instanton is glued to a singular point on X_∞ . The purpose of this paper is to clarify the situation, namely, we get the following theorem stated in terms of the above notation.

THEOREM. *Assume that the sequence (X_i, g_i) consists of Einstein-Kähler surfaces. If we fix a sufficiently small constant $\delta > 0$, then for sufficiently large j , the geodesic ball $B(x_{a,j}; \delta)$ in X_j is diffeomorphic to a cyclic quotient of an ALE gravitational instanton.*

REMARK. In the 4-dimensional case, for a compact Einstein manifold X the curvature integral $\int_X |R|^2 = \text{const } \chi(X)$ is a topological invariant.

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1. Preparation from analysis. Let M be a complete n -dimensional ($n \geq 3$) Riemannian manifold with a fixed point $o \in M$. For $0 < r_1 < r_2$ we denote $B(o; r_2) \setminus B(o; r_1)$ by $D(r_1, r_2)$. We assume that there is a domain $D = D(r_0, r_\infty)$ in M with $0 \leq r_0 < r_\infty$ which satisfies the following conformally invariant conditions:

$$\left\{ \int_D |v|^{2\gamma} \right\}^{1/\gamma} \leq S \int_D |\nabla v|^2 \quad \text{for all } v \in C_c^1(D),$$

$$\text{vol}(D(r_1, r_2)) \leq V r_2^n \quad \text{for all } r_0 \leq r_1 \leq r_2 \leq r_\infty$$

with some positive constants S , V and $\gamma = n/(n-2)$. Let u be a non-negative function defined on D which satisfies

$$\Delta u \geq -fu \quad \text{on } D$$

with a non-negative function f . Then we have the following lemmas. The proofs, which are essentially the same as those for the corresponding lemmas in [4; §4], are omitted.

LEMMA 1. Suppose $f \in L^{n/2}$, and $u \in L^p$ for some $p \in [p_0, p_1]$ where $p_0 > 1$. Then $u \in L^q$ for all $q \geq p$, and there exists $\varepsilon_1 = \varepsilon_1(S, V, p_0, p_1) > 0$ such that if

$$\int_{D(r, 8r)} f^{n/2} \leq \varepsilon_1 \quad \text{with } r_0 \leq r < 8r \leq r_\infty,$$

then we have

$$\left\{ \int_{D(2r, 4r)} u^{p\gamma} \right\}^{1/\gamma} \leq C_1 r^{-2} \int_{D(r, 8r)} u^p,$$

where $C_1 = C_1(S, V, p_0)$. Moreover if $r_0 = 0$ and

$$\int_{B(o; 2r)} f^{n/2} \leq \varepsilon_1 \quad \text{with } 2r \leq r_\infty,$$

then it holds that

$$\left\{ \int_{B(o; r)} u^{p\gamma} \right\}^{1/\gamma} \leq C_1 r^{-2} \int_{B(o; 2r)} u^p.$$

LEMMA 2. Suppose $f \in L^{n/2}$, and $u \in L^p$ for some $p \in [p_0, p_1]$ where $p_0 > \gamma$. Then there exists $\varepsilon_2 = \varepsilon_2(S, V, p_0, p_1) > 0$ such that if

$$\int_D f^{n/2} \leq \varepsilon_2,$$

then it holds that for $r_0 \leq r_1 < 2r_1 < r_2 < 2r_2 \leq r_\infty$

$$\begin{aligned} \int_{D(2r_1, r_2)} u^p &\leq C_2 \int_{D(r_1, 2r_1) \cup D(r_2, 2r_2)} u^p, \\ \int_{D(r_1, r_2)} u^p &\leq C_2 \max \left\{ \left(\frac{r_0}{r_1} \right)^{\varepsilon_3}, \left(\frac{r_2}{r_\infty} \right)^{\varepsilon_3} \right\} \int_D u^p, \end{aligned}$$

where $C_2 = C_2(S, V, p_0)$, $\varepsilon_3 = \varepsilon_3(S, V, p_0) > 0$.

LEMMA 3. If $f \in L^q$ for some $q > n/2$, $u \in L^p$ for some $p > 1$, and if

$$\int_{D(r, 8r)} f^q \leq A r^{-(2q-n)}$$

with some constant A for any r such that $r_0 \leq r < 8r \leq r_\infty$, then we have

$$\sup_{D(2r, 4r)} u^p \leq C_3 r^{-n} \int_{D(r, 8r)} u^p,$$

where $C_3 = C_3(A, S, V, p, q)$. Moreover if $r_0 = 0$ and

$$\int_{B(o; 2r)} f^q \leq A r^{-(2q-n)},$$

then

$$\sup_{B(o; r)} u^p \leq C_3 r^{-n} \int_{B(o; 2r)} u^p.$$

Let (M, g) be an n -dimensional Einstein manifold. Then applying the Weitzenböck formula we get

$$\Delta |R| \geq -C_4 |R|^2.$$

Moreover, we have the following inequality using Yau's trick. For the proof see [2], [4] and [9].

LEMMA 4. *There exist positive constants $\delta = \delta(n)$ and $C_5 = C_5(n)$ such that*

$$\Delta |R|^{1-\delta} \geq -C_5 |R|^{2-\delta}.$$

If $n=4$ or if (M, g) is Kähler, then we can take $\delta = 4/(n+2)$.

One can show the following lemma via L^2 -Hodge theory.

LEMMA 5. *Let (X, g) be an n -dimensional ($n \geq 4$), complete, non-compact, Ricci-flat, ALE orbifold. Then its first cohomology group $H^1(X; \mathbf{R})$ vanishes.*

Here we recall the existence theorem of Ricci-flat Kähler metrics on open Kähler orbifolds in [3], which is stated in the case of manifolds but whose proof works equally for orbifolds.

DEFINITION. A complete n -dimensional Riemannian orbifold (X, g) is said to be of $C^{k, \alpha}$ -asymptotically flat geometry if for each point $p \in X$ with distance from a fixed point o in X , there exists a quasi-coordinate map $\phi: B^n \rightarrow X$ centered at p from the unit ball B^n in the Euclidean space (i.e. ϕ gives a local uniformization and $\phi(0) = p$), such that with respect to the standard coordinates $x = (x^1, x^2, \dots, x^n)$ of the Euclidean space it satisfies the following conditions:

(i) If we write $\phi^*g = \sum g_{ij}(x) dx^i dx^j$, then the matrix $(r^2 + 1)^{-1}(g_{ij})$ is bounded

from below by a constant positive matrix independent of p .

- (ii) The $C^{k,\alpha}$ -norms of $(r^2 + 1)^{-1}g_{ij}$, as functions in x , are uniformly bounded.

On such an orbifold we can define the Banach space $C^{k,\alpha}_\delta$ of weighted $C^{k,\alpha}$ -bounded functions: The norm of a function $u \in C^{k,\alpha}_\delta$ is given by the supremum of the $C^{k,\alpha}$ -norms of $(r^2 + 1)^{\delta/2}u$ with respect to the coordinates x .

THEOREM C. *Let (X, ω) be an n -dimensional ($n \geq 2$) complete open Kähler orbifold of $C^{k,\alpha}$ -asymptotically flat geometry with $k \geq 2$, $0 < \alpha < 1$. Assume that the singularities sit in a compact set and there exists a barrier function ρ . If X admits a Ricci-flat volume form V such that $\omega^n = e^f V$ with $f \in C^{k+2,\alpha}_\delta$ and $\delta > 0$, then X admits a complete Ricci-flat Kähler metric asymptotically equal to ω .*

Here a barrier function ρ means that outside a compact set ρ satisfies the following conditions:

- (i) ρ is compatible with the distance function d from o ; that is, there exists a positive constant c_1 such that $c_1 d \leq \rho \leq c_1^{-1} d$.
- (ii) The function $\rho^{-\delta}$ belongs to $C^{k+2,\alpha}_\delta$.
- (iii) There exists a positive constant c_2 such that

$$\square \rho^{-\delta} \leq -c_2 \rho^{-2-\delta}.$$

- (iv) There exists a positive constant c_3 such that for any positive number K and sufficiently large d

$$\begin{aligned} (\omega + \sqrt{-1} \partial \bar{\partial} K \rho^{-\delta})^n &\leq (1 - c_3 K \rho^{-2-\delta}) \omega^n, \\ (\omega + \sqrt{-1} \partial \bar{\partial} - K \rho^{-\delta})^n &\geq (1 + c_3 K \rho^{-2-\delta}) \omega^n. \end{aligned}$$

2. Einstein manifolds. Let (X_j, g_j) be a sequence of Einstein manifolds which enjoys the properties stated in Theorem A. Then by [5] we have the Sobolev inequality on (X_j, g_j) with uniform Sobolev constants, and the following proposition holds. For the proof see [1], [8].

PROPOSITION 1. *There exist constants ρ , C_6 and ε_4 such that if*

$$\int_{B(x; 2r)} |R_{g_j}|^{n/2} \leq \varepsilon_4$$

with $2r \leq \rho$, then we have

$$\sup_{B(x; r)} |R_{g_j}| \leq C_6 r^{-2} \left\{ \int_{B(x; 2r)} |R_{g_j}|^{n/2} \right\}^{2/n}.$$

Now we take a positive constant $r_\infty < \rho$ sufficiently small, so that we can assume that for all a

$$\sup_{B(x_a, j; r_\infty)} |R_{g_j}|^2 = |R_{g_j}|^2(x_{a, j}) \rightarrow \infty \quad \text{as } j \rightarrow \infty$$

and

$$\int_{B(x_a, r_\infty)} |R_{g_\infty}|^{n/2} \leq \frac{\varepsilon}{2}$$

with a positive number $\varepsilon \leq \varepsilon_4/2$ to be determined later. From now on we fix an arbitrary singular point x_a and look at the blowing up process. Since (X_j, g_j) converges to (X_∞, g_∞) in C^∞ -topology except at the singular points, for sufficiently large j we can find a positive number $r_0 = r_{0, j}$ such that

$$\int_{D(r_0, r_\infty)} |R_{g_j}|^{n/2} = \varepsilon,$$

where we denote the subset $B(x_a, j; r_2) \setminus B(x_a, j; r_1)$ in X_j by $D(r_1, r_2)$. Then we get

$$r_0 \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

PROPOSITION 2. *There is a subsequence $\{k\} \subset \{j\}$ such that the sequence of pointed Einstein manifolds $((X_k, r_0^{-2}g_k), x_{a, k})$ converges to $((Y, h), y_\infty)$ in the pointed Hausdorff distance, where (Y, h) is a complete, non-compact, Ricci-flat, non-flat n -orbifold only with finitely many isolated singular points. (Y, h) is ALE of order $n-1$ in general, and of order n if $n=4$ or (Y, h) is Kähler. The convergence is actually in C^∞ -topology except at the singular points.*

The proof is the same as that of Theorem A. We refer to [1], [8] and [4].

Thus we know that for large $1 < K_1 < K_2$ the two subsets $D(K_1 r_0, K_2 r_0)$ and $D(K_2^{-1} r_\infty, K_1^{-1} r_\infty)$ in X_k are very close to portions of flat cones $\mathbf{R}^n/\Gamma_0$ and $\mathbf{R}^n/\Gamma_\infty$, respectively. To show that $\Gamma_0 = \Gamma_\infty$ and $D(K_1 r_0, K_2^{-1} r_\infty)$ is also close to a portion of the flat cone, we need the following curvature estimate.

PROPOSITION 3. *There exist positive constants C_7 and ε_5 such that for $4r_0 \leq r < 4r \leq r_\infty$ it holds that*

$$r^2 |R_{g_j}| \leq C_7 \max \left\{ \left(\frac{r_0}{r} \right)^{\varepsilon_5}, \left(\frac{r}{r_\infty} \right)^{\varepsilon_5} \right\}.$$

PROOF. First apply Lemma 1 to the equation $\Delta |R| \geq -C_4 |R|^2$ on $R = R_{g_j}$, assuming $C_4^{n/2} \varepsilon \leq \varepsilon_3$. Then we get that for $2r_0 \leq r < 2r \leq r_\infty$

$$\int_{D(r, 2r)} |R|^{n/2} \leq A r^{-(2q-n)}$$

with a constant A and $q = \gamma n/2$. Next we apply Lemmas 2 and 3 to the equation $\Delta |R|^{1-\delta} \geq -C_5 |R|^{2-\delta}$ with $p = (1-\delta)^{-1} n/2 > \gamma$. If $C_3^{n/2} \varepsilon \leq \varepsilon_2$, we get that for $4r_0 \leq r < 4r \leq r_\infty$

$$r^2 |R_{g_j}| \leq C_3^{2/n} \left\{ \int_{D(r/2, 4r)} |R|^{n/2} \right\}^{2/n} \leq C_3^{2/n} (3C_2 2^{\varepsilon_3})^{2/n} \max \left\{ \left(\frac{r_0}{r} \right)^{\varepsilon_5}, \left(\frac{r}{r_\infty} \right)^{\varepsilon_5} \right\}$$

with $\varepsilon_5 = 2\varepsilon_3/n$. We choose $\varepsilon = \min\{\varepsilon_3 C_4^{-n/2}, \varepsilon_2 C_5^{-n/2}, \varepsilon_4/2\}$, and the proof is complete.

Once we get the curvature estimate, we can construct coordinates as in the proof of the existence theorem of coordinates at infinity [4]. We need only minor changes, so we omit the proof of the following proposition.

PROPOSITION 4. *If one takes $1 < K_1 < K_2$ sufficiently large, then the subset $D(K_1 r_0, K_2^{-1} r_\infty)$ is close to a portion of a flat cone $\mathbb{R}^n/\Gamma$ for large j .*

Thus if (Y, h) has no singularity, then the ball $B(x_{a,k}; r_\infty)$ is diffeomorphic to the smooth manifold Y which bubbles out of X_k .

If (Y, h) has a singular point y_s , then we choose a sufficiently small number r'_∞ and the corresponding point $x_{s,k}$ in X_k so that

$$\int_{B(y_s, r'_\infty)} |R_h|^{n/2} \leq \frac{\varepsilon}{2}$$

$$\sup_{B(x_{s,k}; r_0 r'_\infty)} |R_{g_k}|^2 = |R_{g_k}|^2(x_{s,k}) \rightarrow \infty.$$

Choose $r'_0 = r'_{0,k}$ so that

$$\int_{D'(r_0 r'_0, r_0 r'_\infty)} |R_{g_k}|^{n/2} = \varepsilon,$$

with $D'(r_1, r_2) = B(x_{s,k}; r_2) \setminus B(x_{s,k}; r_1)$, and consider a sequence of pointed Einstein manifolds $((X_k, (r_0 r'_0)^{-2} g_k), x_{s,k})$. Then we have the same situation as before, and we get a complete, non-compact, Ricci-flat, non-flat, ALE n -orbifold (Y', h') only with finitely many isolated singular points. In the same way we can show that the neck is diffeomorphic to a flat cone. If (Y', h') again has a singular point, we repeat the argument. We also apply the same process at every singular point which appears at each repeated step. Since each singular point contributes at least ε to the curvature integral $\int |R|^{n/2}$, the process terminates in finite steps. In this way we get a picture of the small ball $B(x_{a,j}; r_\infty)$.

DEFINITION. Let X and Y be complete, non-compact, ALE n -orbifolds such that X has a point x which has a neighborhood diffeomorphic to $B(0; R)/\Gamma$ and Y has an end which is diffeomorphic to $(\mathbb{R}^n \setminus B(0; R))/\Gamma$ with the same $\Gamma \subset O(n)$. Since $\partial B(0; R)/\Gamma$ is diffeomorphic to $\partial(\mathbb{R}^n \setminus B(0; R))/\Gamma$, we can glue X and Y along them and get a new ALE n -orbifold $X \# Y$, which we call an IS-connected sum of X and Y .

around x and the end of Y .

THEOREM 1. *The small ball $B(x_{a,j}; r_\infty)$ in X_j corresponding to a singular point x_a of the limit orbifold X_∞ is diffeomorphic to an IS-connected sum of finite number of complete, non-compact, Ricci-flat, non-flat, ALE n -orbifolds only with finitely many isolated singular points, where all singular points are glued to the ends and we end up with an ALE manifold.*

REMARK. We may also use the following gap theorem to show that the process terminates in finite steps.

THEOREM 2. *Let (X, g) be an n -dimensional ($n \geq 4$), complete, non-compact, Ricci-flat Riemannian orbifold which satisfies*

$$\left\{ \int v^{2\gamma} \right\}^{1/\gamma} \leq S \int |\nabla v|^2 \quad \text{for all } v \in C_c^1(X)$$

with a constant $S > 0$. Then there exists a constant $\varepsilon_6 = \varepsilon_6(n, S) > 0$ such that the inequality

$$\int_X |R|^{n/2} \leq \varepsilon_6$$

implies that (X, g) is the Euclidean space.

PROOF. Apply Lemma 1.

3. Einstein Kähler surfaces. In this section we assume that all manifolds (X_j, g_j) are Einstein-Kähler surfaces. Since the limit space X_∞ is an orbifold, there is a neighborhood U of the singular point x_a which is biholomorphic to a quotient B/Γ of the unit ball $B \subset \mathbb{C}^2$ with a finite subgroup $\Gamma \subset U(2)$ acting freely on $\mathbb{C}^2 \setminus \{0\}$. Let $\det: U(2) \rightarrow S^1$ be the group homomorphism defined by the determinant. Then the image $\det(\Gamma)$ is a finite cyclic group, say, $\mathbb{Z}_m$. Then U has a branched $\mathbb{Z}_m$ -covering: $\tilde{U} \rightarrow U$ with a branch point x_a such that $\tilde{U}$ has trivial canonical line bundle $K_{\tilde{U}}$. Namely, set $\tilde{\Gamma} = (\ker \det) \cap \Gamma \subset SU(2)$. Then we have a natural projection $\tilde{U} = B/\tilde{\Gamma} \rightarrow U$ and the non-vanishing holomorphic 2-form $\omega = dz^1 \wedge dz^2$ descends to $\tilde{U}$, where (z^1, z^2) is the standard coordinates in $\mathbb{C}^2$. We have the corresponding result on $x_{a,j} \in X_j$ for large j .

PROPOSITION 5. *There exists a positive constant δ such that for j large there is a smooth $\mathbb{Z}_m$ -covering: $\tilde{U}_j \rightarrow U_j \supset B(x_{a,j}; \delta)$ with $\tilde{U}_j$ having topologically trivial canonical line bundle $K_{\tilde{U}_j}$.*

PROOF. We may assume that the domain $U \subset X_\infty$ has smooth boundary ∂U . Then there exists a sequence of neighborhoods $U_j \subset X_j$ of $x_{a,j}$ which have smooth boundaries $\partial U_j = F_j(\partial U)$. We take δ so small that $B(x_{a,j}; \delta) \subset U_j$. Then it is sufficient to show that for large j there are sections θ_j of $K_{\tilde{U}_j}^{\otimes m}$ on U_j such that

$$C_8^{-1} \leq |\theta_j| \leq C_8, \quad \text{and} \quad |\nabla \theta_j| \leq C_9$$

with positive constants C_8, C_9 .

Define an operator $\square = \square_j$ acting on the space of sections of $K_{X_j}^{\otimes m}$ by

$$\square = -\bar{\partial}^* \bar{\partial} = \text{tr } \nabla' \nabla'',$$

where we decompose the covariant differentiation $\nabla = \nabla' + \nabla''$ into $(1, 0)$ - and $(0, 1)$ -parts. Let ψ be the local holomorphic uniformization $\psi: B \rightarrow B/\Gamma \cong U$ and η be a radial cut-off function on B such that $\eta = 0$ on $B(0; 1/3)$ and $\eta = 1$ on $B \setminus B(0; 2/3)$. Through ψ the section $\eta \omega^{\otimes m}$ of $K_B^{\otimes m}$ defines a section of $K_U^{\otimes m}$, which we still denote by $\eta \omega^{\otimes m}$. For j large we define sections $\theta_0 = \theta_{0,j}$ of $K_{X_j}^{\otimes m}$ on U_j by $\theta_{0,j} = \text{proj}(F_j^{-1})^* \eta \omega^{\otimes m}$, where $\text{proj} = \text{proj}_j$ is the projection map for tensors to $K_{X_j}^{\otimes m}$. (Note that the maps $F_j: X_\infty \setminus S \rightarrow X_j$ need not be holomorphic, but become closer and closer to be holomorphic as j tends to ∞). We solve the following equation for a section $\theta = \theta_j$ of $K_{X_j}^{\otimes m}$ on U_j :

$$\square \theta = 0 \quad \text{and} \quad \theta|_{\partial U_j} = \theta_0|_{\partial U_j}.$$

Then θ satisfies

$$\Delta \theta = \text{tr } \nabla \nabla \theta = -2me_j \theta.$$

Set $\theta' = \theta - \theta_0$. Then θ' has vanishing boundary value and satisfies

$$\Delta \theta' = -2me_j \theta' + \zeta$$

with $\zeta = \zeta_j$ on which we have good control. We have

$$\begin{aligned} \lambda \int |\theta'|^2 &\leq \int |\nabla \theta'|^2 \leq \int |\nabla \theta'|^2 = - \int (\theta', \Delta \theta') = 2me_j \int |\theta'|^2 - \int (\theta', \zeta) \\ &\leq 2me_j \int |\theta'|^2 + \left(\int |\theta'|^2 \right)^{1/2} \left(\int |\zeta|^2 \right)^{1/2} \end{aligned}$$

with the first eigenvalue $\lambda = \lambda_j$ of the Laplacian acting on functions on U_j with the Dirichlet condition. If we choose U , hence U_j , sufficiently small so that $\lambda \geq 2m|e_j| + 1$, we get L^2 -estimates for θ' and $\nabla \theta'$, hence, for θ and $\nabla \theta$. We apply Lemma 3 to the inequality $\Delta |\theta| \geq -2m|e_j| |\theta|$, and get C^0 -estimate for θ .

As for C^1 -estimate we differentiate the equation on θ and get the following equations:

$$\begin{aligned} \Delta |\nabla' \theta|^2 &= 2|\nabla \nabla' \theta|^2 + 2e_j |\nabla' \theta|^2, \\ \Delta |\nabla'' \theta|^2 &= 2|\nabla \nabla'' \theta|^2 + 2(-4m+1)e_j |\nabla'' \theta|^2, \\ \Delta |\nabla \theta|^2 &\geq 2|\nabla \nabla \theta|^2 - 2(4m-1)|e_j| |\nabla \theta|^2, \\ \Delta |\nabla \theta| &\geq -(4m-1)|e_j| |\nabla \theta|. \end{aligned}$$

Then again applying Lemma 3, we get C^1 -estimate away from boundaries. As for one near boundaries, we have good control on the smoothness of the boundaries, the boundary values and the equations. We also have C^0 -estimate for θ . So there is no trouble in getting C^∞ -estimate on $U_j \setminus B(x_{a,j}; r)$ for any fixed $r > 0$.

Now consider the sequence $\{\text{proj } \psi^* F_j^* \theta_j\}$ on $B \setminus \{0\}$ which has uniform C^∞ -estimate away from the origin 0. Thus it has a convergent subsequence with limit, say, $\tilde{\theta}$ defined on $B \setminus \{0\}$. $\tilde{\theta}$ satisfies the equation $\square \tilde{\theta} = 0$ and has C^1 -estimate, so it extends to a smooth solution of the equation across the origin. It must coincide with the unique solution $\omega^{\otimes n}$. Hence the sequence $\{\text{proj } \psi^* F_j^* \theta_j\}$ itself converges to $\omega^{\otimes n}$, and there is a positive constant C_{10} such that for fixed $r > 0$ we have that for large $j \geq j(r)$

$$|\theta_j| \geq C_{10} \quad \text{on } U_j \setminus B(x_{a,j}; r).$$

By Theorem 1 there exists a constant C_{11} such that every point in $B(x_{a,j}; r)$ can be connected to the boundary $\partial B(x_{a,j}; r)$ with a curve of length at most $C_{11}r$. Thus for $j \geq j(r)$ with $r = C_{10}/2C_9C_{11}$ we have $|\theta_j| \geq C_{10}/2$ on U_j .

REMARK. One can also show that $K_{U_j}^{\otimes m}$ is complex analytically trivial for large j .

From now on we work on the covering space $\tilde{U}_j$, and denote it simply by U_j . Then we have $m=1$. We made a trivialization θ of K_{U_j} with uniform C^1 -estimate. Thus if we conformally change it, the triviality is preserved in the process of bubbling out of complete, Ricci-flat, ALE orbifold Kähler surfaces. Hence the local fundamental groups of the singular points and the fundamental group at infinity of each bubble are contained in $SU(2)$.

PROPOSITION 6. *Let (X, g) be a complete, Ricci-flat, ALE orbifold Kähler surface. If its canonical line bundle K_X is topologically trivial, then (X, g) is hyper-Kähler.*

PROOF. By assumption K_X is flat and defines an element in $H^1(X; S^1)$. The exact sequence

$$H^1(X; \mathbf{R}) \rightarrow H^1(X; S^1) \rightarrow H^2(X; \mathbf{Z})$$

and Lemma 5 imply that the topologically trivial K_X has a trivial connection.

Thus our bubbles are all hyper-Kähler. Hence if there is only one bubble coming out, the proof of the main theorem is complete.

THEOREM 3. *If we take $\delta > 0$ sufficiently small, then for sufficiently large j , the geodesic ball $B(x_{a,j}; \delta)$ in X_j is diffeomorphic to a cyclic quotient of an ALE gravitational instanton.*

REMARK. It is likely that $B(x_{a,j}; \delta)$ is biholomorphic to a domain of a cyclic quotient of an ALE gravitational instanton.

We can prove Theorem 3 by applying the following theorem inductively.

THEOREM 4. *Let (X, g) be a complete, hyper-Kähler, ALE orbifold surface which has a singular point o with local fundamental group $\Gamma \subset SU(2)$, and (Y, h) be an ALE gravitational instanton which is biholomorphic to the minimal resolution of C^2/Γ . Then an IS-connected sum $X \# Y$ around o and the end of Y also admits a structure of a complete, hyper-Kähler, ALE orbifold surface.*

PROOF. First fix a Kähler structure (X, ω_1) on X , where ω_1 is its Kähler form. We can take a holomorphic local uniformization $\psi_1: B(0; \delta) \subset C^2 \rightarrow U \ni o$ so that

$$\psi_1^* \omega_1 = \sqrt{-1} \partial \bar{\partial} \phi_1, \quad \phi_1 = |z|^2 + O(|z|^3),$$

$$\psi_1^* \omega_1^2 = 2(\sqrt{-1} dz^1 \wedge d\bar{z}^1)(\sqrt{-1} dz^2 \wedge d\bar{z}^2).$$

Let $\psi_2: C^2 \setminus B(0; K) \rightarrow Y$ be the holomorphic local uniformization of Y at infinity. Then by Kronheimer [6] the Kähler form ω_2 of (Y, h) satisfies the following properties (cf. [3]):

$$\psi_2^* \omega_2 = \sqrt{-1} \partial \bar{\partial} \phi_2, \quad \phi_2 = |z|^2 + O(|z|^{-2}),$$

$$\psi_2^* \omega_2^2 = 2(\sqrt{-1} dz^1 \wedge d\bar{z}^1)(\sqrt{-1} dz^2 \wedge d\bar{z}^2).$$

For sufficiently small positive numbers δ_1, δ_2 , by the map $\psi(z) = z/\delta_1 \delta_2$ we identify two subsets $\psi_1(D(\delta_1, 4\delta_1)) \subset X$, $\psi_2(D(\delta_2^{-1}, 4\delta_2^{-1})) \subset Y$, and get an orbifold surface $Z = X \# Y$. In this construction the parallel holomorphic 2-forms on X and Y are glued to give a holomorphic 2-form on Z . We define a Kähler metric ω on Z as follows:

$$\omega = \begin{cases} \omega_1 & \text{on } X \setminus \psi_1(B(0; 4\delta_1)) \\ \sqrt{-1} \partial \bar{\partial} \{ \eta_{4\delta_1} \phi_1 + (1 - \eta_{4\delta_1})(\delta_1 \delta_2)^2 \psi^* \phi_2 \} & \text{on } \psi_1(D(\delta_1, 4\delta_1)) \\ (\delta_1 \delta_2)^2 \omega_2 & \text{on } Y \setminus \psi_2(C^2 \setminus B(0; \delta_2^{-1})), \end{cases}$$

where $\eta_\delta(z) = \eta(z/\delta)$ is a cut-off function. Since $\phi_1 - |z|^2$ and $(\delta_1 \delta_2)^2 \psi^* \phi_2 - |z|^2$ are small on $\psi_1(D(\delta_1, 4\delta_1))$, it is easy to see that ω actually defines a Kähler metric on Z .

By assumption there is a coordinate $\psi_\infty: R^4 \setminus B(0; K) \rightarrow X$ at the infinity of X such that

$$\psi_\infty^* g_{ij} = \delta_{ij} + O(|x|^{-4}).$$

Then it is easy to see that $\rho = |x|$ makes a barrier function on X , hence on Z . Thus (Z, ω) satisfies the assumption of Theorem C, that is, Z admits a complete, Ricci-flat, ALE orbifold Kähler metric. Z is hyper-Kähler, since the holomorphic 2-form on Z is easily seen to be parallel.

Now we prove Theorem 3. Assume that the blowing up process of orbifold singular points terminates in l steps. Then the bubbles coming out at the l -th step are all smooth ALE gravitational instantons. So they are diffeomorphic to the minimal resolutions of

C^2/Γ , $\Gamma \subset SU(2)$. We replace their structures by those coming from minimal resolutions. Then by Theorem 4 we can glue them to the bubbles of the $(l-1)$ -th step, and get smooth ALE gravitational instantons. Repeating this argument we finally get a smooth ALE gravitational instanton which is given by all bubbles glued. This implies Theorem 3.

For examples of bubbling out of ALE gravitational instantons we refer to [7].

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